Cosmology 2 TA1: recap of Friedmann equations

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1 Isotropic and Homogeneous universe

One of the first assumption in cosmology is that the universe is isotropic (i.e. the universe looks the same in every line of sight) and homogeneous (the properties of the universe do not depend on the particular point in space we are). Another way of saying, is that the universe is rotation and translation invariant (but critically, not invariant under time translations, i.e. the universe looks different at different points in time). How much is this assumption justified? After all, one can just look around to blatantly falsify this assumption.

But if we look at the very early universe through the CMB, Fig. 1, or we look at very large scales, Fig. 2, this approximation does not look bad any more. In this course we will deal mostly with early universe physics, so let's put our focus on the CMB. In a nutshell, the CMB is the Cosmic Microwave Background radiation permeating our Universe, whose origin dates back to when the universe was approximately 380000 years old. The spectrum of this radiation is one of the best example of black body radiation known, and figures like the ones in Fig. 1 represent the temperature of this black body from different line of sights. These temperature differences are connected with overdensities or underdensities present at the formation of the CMB, hence making the CMB a snapshot of how the universe looked like back then.

At first, if one looks at the left panel of Fig. 1, one sees that the CMB spectrum does not look isotropic at all; there is clearly a dipole present. This dipole contribution is thought to come from the movement of the Earth, Sun and Milky way with respect to a reference frame where the CMB does indeed look isotropic. We call this frame the comoving frame, and it is only with respect to this frame that the universe looks isotropic[1]. Strictly speaking, humans on Earth are not comoving observer, due to the peculiar motion of Earth, Sun etc. with respect to the overall expanding universe[2].

Once in this comoving frame, one has to look the CMB with instruments capable of 10^{-5} temperature determination precision to see meaningful deviations from perfect isotropy. These temperature fluctuations are related to density fluctuations. If we split

$$\rho(\vec{x},t) = \bar{\rho}(t)(1+\delta(\vec{x},t)) \ , \tag{1.1}$$

where $\bar{\rho}$ is the average density at comoving time t, also called background density, and δ is the fluctuation over such a background, then during the CMB formation it is true that $\delta \ll 1$. What this implies is that we can describe the early universe as a perturbed homogeneous and isotropic one, where homogeneous and isotropic is the zeroth order approximation. We will focus on just this zeroth order in the following, but later in the course we will see how to go beyond this order.

2 The Friedmann equations

Upon an isotropic and homogeneous universe, the most general line element is the Friedmann-Robertson-Walker (FRW) one, which in comoving coordinates reads

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - a^2(t)\left(\frac{\mathrm{d}r^2}{1-\kappa r^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)\right) \ . \tag{2.1}$$

[1]What we can determine from our solar system is that the universe looks isotropic, but actually we cannot really say whether it is also homogeneous. To arrive at such conclusion, one has to invoke the Copernican principle, which states that it is unlikely that we live in a very special region of the universe. In this context, we can say it is very unlikely we live in just one of the few regions where the universe looks isotropic. Once we accept this principle, then isotropy from every space point implies homogeneity.

[2]Be aware that general relativity does NOT claim that there is no privileged reference frame or choice of coordinates. The correct claim is that physics is invariant over such a choice, which is a different statement.

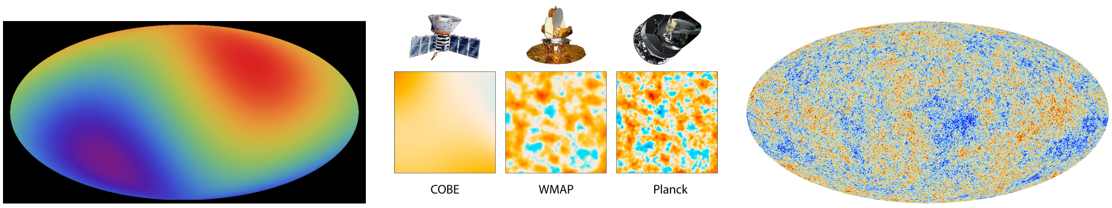

Figure 1: On the left, CMB snapshot without the dipole contribution subtracted. On the center, evolution of the representations of the CMB. On the right, the up-to-date figure from Planck 18.

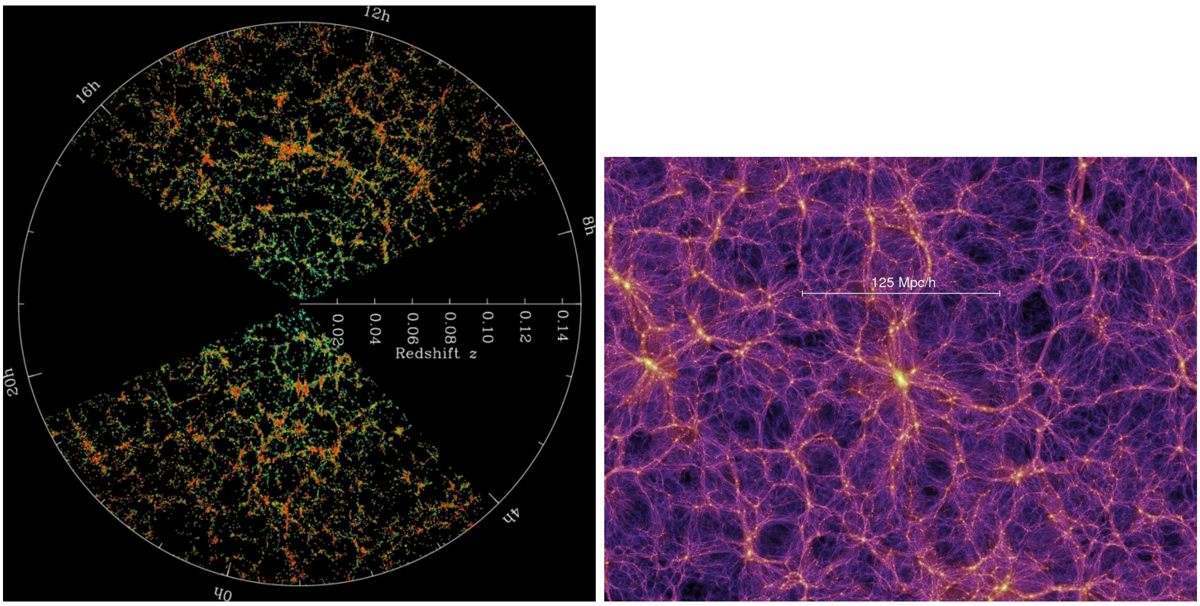

Figure 2: On the left, galaxies distributed in redshift on the sky, from SDSS. On the right, example of the cosmic web, from millennium simulation. On large scales, the universe looks indeed homogeneous and isotropic.

κ is the curvature parameter; homogeneity and isotropy in space implies that the spatial Ricci curvature should be a constant, ${}^{(3)}R \propto \kappa$; we have a flat, spherical/closed or hyperbolic/open universe for $\kappa = 0$, $\kappa > 0$ or $\kappa < 0$ respectively. Our universe seems best described as a flat one, $\kappa \simeq 0$.

We have the freedom to set the scale factor at the present time as $a(t_0) = 1$, thanks to the invariance under the rescaling

$$a \to \lambda a \ , \ r \to r/\lambda \ , \ \kappa \to \lambda^2 \kappa \ . \tag{2.2}$$

Notice that the sign of κ remains unchanged. Be aware that in cosmology, it is customary to label with a zero subscript quantities which refer to the present time, rather than some initial time. With this choice, comoving distances x correspond to physical distance today, since

$$x^{\mathrm{phys}} = a(t)x \ . \tag{2.3}$$

We determined the metric, which enters the LHS of Einstein equations (in natural units)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \ ; \tag{2.4}$$

the proper stress-energy tensor obeying homogeneity and isotropy is the perfect fluid one, which in covariant formulation is

$$T_{\mu\nu} = -\bar{P}(t) g_{\mu\nu} + (\bar{P}(t) + \bar{\rho}(t)) u_\mu u_\nu \ , \tag{2.5}$$

where u^μ is the four velocity of the observer. A comoving observer has $u^\mu = (1, 0, 0, 0)$. It is important to remark that $\bar{\rho}$, $\bar{P}$ depend on the time t alone (no space dependence).

From this, we obtain the Friedmann equations:

$$\left(\frac{\dot{a}}{a}\right)^2 =: H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \ , \tag{2.6}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{P}) + \frac{\Lambda}{3} \ ; \tag{2.7}$$

the third Friedmann equation, also called continuity equation, can be derived from the previous two or also from the zero component of the continuity equation $T^{\mu\nu}{}_{;\nu} = 0$, and reads

$$\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P}) = 0 \ . \tag{2.8}$$

Notice that κ appears in equations always in the combination κ/a^2, which is invariant under the rescaling Eq. (2.2).

Different components of the universe are characterized by a different equation of state $\bar{P} = w_{\mathrm{s}}\bar{\rho}$, in particular $w_{\mathrm{s}} = 0$ for matter (non-relativistic matter has negligible pressure with respect to energy density) and $w_{\mathrm{s}} = 1/3$ for radiation (the usual result for an ultra-relativistic gas).

It is worth stopping for a second to consider the case $w_{\mathrm{s}} = -1$, a negative pressure fluid. If we split the density in possible different components

$$\rho = \rho_\Lambda + \sum_i \rho_i \ , \tag{2.9}$$

where ρ_Λ is the component with equation of state $w_{\mathrm{s}} = -1$ and ρ_i with $w_{\mathrm{s}}^i \neq -1$, we have, from Eq. (2.5)

$$T_{\mu\nu} = \rho_\Lambda g_{\mu\nu} + \sum_i T_{\mu\nu}^{(i)} \ . \tag{2.10}$$

We see that the ρ_Λ component is degenerate with the cosmological constant term Λ in the Einstein equation. This means that we can define an effective cosmological constant

$$\Lambda^{\mathrm{eff}} = \Lambda + 8\pi G \rho_\Lambda \ . \tag{2.11}$$

Notice that, from Eq. (2.8), $\dot{\bar{\rho}} = 0$, hence constant. From a physical point of view, Λ is a parameter entering the GR equations, it can be interpreted as a parameter which defines the geometrical property of the space-time we live in. ρ_Λ, on the other hand, comes from the energy-density budget of the universe, in particular its origin can come from vacuum expectation values of standard model (or beyond standard model) fields. In the following, we will not make such a distinction and we will drop the $^{\rm eff}$ superscript.

Using the equations of states for the various density components of the universe, we can express the first Friedmann equation in terms of the various density parameters today

$$\Omega_{j,0} := \frac{8\pi G}{3H_0^2}\bar{\rho}_{j,0} \ , \tag{2.12}$$

where again we used 0 as a subscript for quantities evaluated at the present time, and the index j labels the various components (r = radiation, m=matter, Λ = dark energy, $\bar{\rho}_{\Lambda,0} = \Lambda/8\pi G$). Notice that, if the total density of the universe is exactly

$$\rho_{\rm c} := \frac{3H_0^2}{8\pi G} \ , \tag{2.13}$$

where $\rho_{\rm c}$ is called critical density, it would imply a flat universe, $\kappa = 0$. Using the third Friedmann equations implementing the equation of state, we have

$$\bar{\rho}(t) = \bar{\rho}(t_0)\left(\frac{a(t)}{a(t_0)}\right)^{-3(1+w_{\rm s})} \ . \tag{2.14}$$

With this, the first Friedmann equation becomes

$$H^2 = H_0^2\left(\Omega_{\rm r,0}\left(\frac{a}{a(t_0)}\right)^{-4} + \Omega_{\rm m,0}\left(\frac{a}{a(t_0)}\right)^{-3} + \Omega_{\kappa,0}a^{-2} + \Omega_{\Lambda,0}\right) \ , \tag{2.15}$$

where $\Omega_{\kappa,0}/a^2 = -\kappa/(H_0^2 a^2)$ parametrizes how far are we from a flat universe case. Our universe so far seems compatible with $\Omega_{\rm r,0} \leq 10^{-5}$, $\Omega_{\kappa,0} \sim 0$, $\Omega_{\rm m,0} \sim 0.3$, $\Omega_{\Lambda,0} \sim 0.7$.

3 Distances and horizons

In this section, it will prove to be useful to write the metric using the conformal time $\mathrm{d}\eta = \mathrm{d}t\,/a(t)$ instead of the comoving time, and the comoving coordinate w; defining

$$f_\kappa(w) = \begin{cases} \dfrac{1}{\sqrt{\kappa}}\sin\!\big(\sqrt{\kappa}w\big) \ , \ \kappa > 0 \ ; \\ w \ , \ \kappa = 0 \ ; \\ \dfrac{1}{\sqrt{-\kappa}}\sinh\!\big(\sqrt{-\kappa}w\big) \ , \ \kappa < 0 \ ; \end{cases} \tag{3.1}$$

we end up (with $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2$)

$$\mathrm{d}s^2 = a^2(\eta)\big(\mathrm{d}\eta^2 - \mathrm{d}w^2 + f_\kappa^2(w)\mathrm{d}\Omega^2\big) \ . \tag{3.2}$$

In this form, for the case $\kappa = 0$, the metric is just conformally equivalent to Minkowski, hence giving the name conformal time to η.

Preliminary: geodesics in FRW. We want first to find the geodesics for a FRW universe. One can easily find the geodesic equations for a generic metric by building the Lagrangian associated,

$$\mathcal{L}(x^\mu, \mathrm{d}x^\mu/\mathrm{d}\lambda) = g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda} = \begin{cases} 0 & \text{for massless particles ,} \\ > 0 & \text{for massive particles .} \end{cases} \tag{3.3}$$

We can use the choice of the geodesic parameter λ to normalize $g_{\mu\nu}u^\mu u^\nu = 1$, $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$,[3] for massive particles, whereas for massless particles we can set $u^\mu = P^\mu$[4], where P^μ is the four-momentum (for massive particles, $P^\mu = mu^\mu$). The geodesic equations then read from Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial\mathcal{L}}{\partial\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}} = \frac{\partial\mathcal{L}}{\partial x^\mu} \ . \tag{3.4}$$

[3]With this choice, λ is the proper time of a particle following the geodesic. For a massless particle, the proper time is always zero, so one cannot use proper time to represent massless particle geodesics.

[4]Notice that this loosely mean that we are using $\lambda \sim \lim_{m\to 0}\tau/m$, where τ is proper time.

Considering geodesics with constant θ, φ, we can focus on only

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\left(a^2 \frac{\mathrm{d}w}{\mathrm{d}\lambda}\right) = 0 \implies \frac{\mathrm{d}w}{\mathrm{d}\lambda} = \frac{C}{a^2} \ ; \ \frac{\mathrm{d}\eta}{\mathrm{d}\lambda} = \pm \frac{\mathrm{d}w}{\mathrm{d}\lambda} \ , \tag{3.5}$$

where the last comes directly from Eq. (3.3), with $\mathrm{d}\theta/\mathrm{d}\lambda = \mathrm{d}\varphi/\mathrm{d}\lambda = 0$. Covariant notions of energy E and three-momentum p associated to a particle geodesic characterized by the four-momentum P^μ as seen by an observer with four-velocity v^μ are[5]

$$E = v_\mu P^\mu \ , \ p^2 = P^\mu P_\mu - (P^\mu V_\mu)^2 \ ; \tag{3.7}$$

for a massless particle, $E = p$. A comoving (massive) observer (in coordinates from Eq. (3.2)) has $v^\mu = (1/a, 0, 0, 0)$, so

$$E \propto \frac{1}{a} \ , \ p \propto \frac{1}{a} \ , \tag{3.8}$$

implying that massless particles energy scales as $1/a$, hence explaining the a^{-4} behaviour of radiation density. This does not hold for massive particles (exercise).

Exploiting the metric, we can derive the expression for the comoving metric distance (which corresponds to the comoving distance for $\kappa = 0$) as

$$\eta = w = \int_0^t \frac{\mathrm{d}t}{a} = \int_0^z \frac{\mathrm{d}z}{H(z)} \ , \tag{3.9}$$

where we defined the cosmological redshift

$$1 + z := \frac{1}{a} \ . \tag{3.10}$$

Redshift is really the red-shift of the waveform of a pulse of light, when emitted from a source at time $t < t_0$ and reaches the observer at time t_0. From Eq. (3.8), using $p \propto \lambda$,

$$\frac{\lambda_\mathrm{o} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}} = \frac{1}{a(t_\mathrm{e})} - 1 = z \ . \tag{3.11}$$

Using equation (2.15), we have

$$w(z) = \frac{1}{H_0 \Omega_{\Lambda,0}^{1/2}} \int_0^z \left(\frac{\Omega_{\mathrm{m},0}}{\Omega_{\Lambda,0}} (1+z')^3 + 1 + \frac{\Omega_{\kappa,0}}{\Omega_{\Lambda,0}} (1+z')^2 \right)^{-1/2} \mathrm{d}z' \ . \tag{3.12}$$

The comoving distance will then be $f_\kappa(w)$.

Angular diameter distance. The comoving distance is not something that we can actually measure. Due to this, in cosmology one usually defines other types of distances. One example is the angular diameter distance. Consider an object with physical transverse length D; then, in analogy with the Euclidean case, one defines, with $\delta\theta$ the angle subtended by the object,

$$D_\mathrm{A} := \frac{D}{\delta\theta} \ ; \tag{3.13}$$

the FRW line element implies the following relation for D

$$D = a(t) f_\kappa(w(t)) \delta\theta \ , \tag{3.14}$$

so that

$$D_\mathrm{A}(z) = \frac{f_\kappa(w(z))}{1+z} \ . \tag{3.15}$$

[5] Notice that we can write a generic 4-momentum as

$$P^\mu = E v^\mu + (g^\mu_\nu - v^\mu v_\nu) P^\nu \ , \tag{3.6}$$

where $g^\mu_\nu - v^\mu v_\nu$ projects on the subspace perpendicular to v^μ.

Luminosity distance. Very similarly, again in analogy with the Euclidean case, one can define the luminosity distance $D_{\rm L}$ via

$$F = \frac{L}{4\pi D_{\rm L}^2} \ , \tag{3.16}$$

where F is the flux (energy per second per area) received from an observer, coming from a source with luminosity L (emitted energy per second in the emitter frame). To write a formula for $D_{\rm L}$, one has to be careful about expansion of the universe effects. Schematically, the luminosity in the emitter frame is

$$L_{\rm e} \sim \frac{\Delta E_{\rm e}}{\Delta t_{\rm e}} \tag{3.17}$$

whereas, when we measure the flux, we see $\Delta E_{\rm o}/\Delta t_{\rm o}$. The energy redshifts as in Eq. (3.8), $\Delta E_{\rm o} = \Delta E_{\rm e} a(t_{\rm e})$; to understand what is $\Delta t_{\rm o}$, suppose that a source located at comoving (not conformal) coordinates x emits a light pulse at $t_{\rm e}$ and at $t_{\rm e} + \Delta t_{\rm e}$; the comoving distance the two pulses travel to the observer are the same, hence

$$x = \int_{t_{\rm e}}^{t_{\rm o}} \frac{{\rm d}t}{a} = \int_{t_{\rm e}+\Delta t_{\rm e}}^{t_{\rm o}+\Delta t_{\rm o}} \frac{{\rm d}t}{a} \implies \int_{t_{\rm o}}^{t_{\rm o}+\Delta t_{\rm o}} \frac{{\rm d}t}{a} - \int_{t_{\rm e}}^{t_{\rm e}+\Delta t_{\rm e}} \frac{{\rm d}t}{a} = 0 \ ; \tag{3.18}$$

for small intervals of time, the previous indeed implies

$$\frac{\Delta t_{\rm o}}{a(t_{\rm o})} = \frac{\Delta t_{\rm e}}{a(t_{\rm e})} \ . \tag{3.19}$$

Moreover, at the time of observation, the light spread out to an area $A = 4\pi f_\kappa^2(w)$, ending up with

$$F = \frac{\Delta E_{\rm o}}{4\pi f_\kappa^2(w)\Delta t_{\rm o}} = \frac{a^2(t_{\rm e})L_{\rm e}}{4\pi f_\kappa^2(w)} \implies D_{\rm L} = f_\kappa(w)(1+z) \ . \tag{3.20}$$

Horizons. The comoving particle horizon is defined as the maximum comoving distance from which a signal in the past can influence an observer at time t. If $\kappa = 0$, it is defined as

$$x_{\rm ph}(t) = \int_0^t \frac{{\rm d}t'}{a(t')} = \int_0^a \left\{\frac{1}{aH}\right\} {\rm d}\ln a \ \ ; \tag{3.21}$$

whether this quantity is finite or not (at any given t) depends on the form of $a(t)$. On the last step, we rewrote $x_{\rm ph}$ by explicitly writing what is called the comoving Hubble radius $x_{\rm H} := 1/(aH)$, often used as a proxy for the sphere an observer at a given time can influence. Conversely, the comoving event horizon is the maximum comoving distance which can be reached by a signal sent at time t. It is

$$x_{\rm eh}(t) = \int_t^\infty \frac{{\rm d}t'}{a(t')} \ ; \tag{3.22}$$

this quantity can be (and for our universe, is) finite, meaning there are regions of space which will never be reached by us, not even in the infinite future.

A Deriving Friedmann equations from GR

In this section, I will use the sign conventions

- Metric: $[S1](+---)$
- Christoffel symbols $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu})$
- Riemann: $R^\mu{}_{\alpha\beta\gamma} = 2[S2](\partial_{[\beta}\Gamma^\mu_{\gamma]\alpha} + \Gamma^\mu_{\sigma[\beta}\Gamma^\sigma_{\gamma]\alpha})$
- Ricci: $R_{\mu\nu} = [S2][S3]R^\alpha{}_{\mu\alpha\nu}$
- Einstein eq.: $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = [S3]8\pi G T_{\mu\nu}$

with $[S1] = [S2] = [S3] = 1$. Late latin indices (i, j, k, l etc.), can assume value 1,2 or 3; greek indices can assume value 0, 1, 2 or 3.

We will use the tetrad (Vierbein) formalism. Rewrite the metric using the tetrad $e^{\hat{\mu}}$

$$\mathrm{d}s^2 = \eta_{\hat{\mu}\hat{\nu}} e^{\hat{\mu}} e^{\hat{\nu}} \ , \tag{A.1}$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is the Minkowski metric, and

$$e^{\hat{0}} = a\,\mathrm{d}\eta =: e^{\hat{0}}_{0}\,\mathrm{d}\eta \ , \ e^{\hat{i}} = e^{\hat{i}}_{i}\,\mathrm{d}x^i \ , \ e^{\hat{i}}_{j} = (a, af_\kappa, af_\kappa \sin\theta)\delta^{\hat{i}}_{j} \ , \ x^i = (w, \theta, \varphi) \ ; \tag{A.2}$$

the idea of the tetrad formalism is to work in a frame where the metric looks locally Minkowskian (thanks to the equivalence principle, this is always possible). We will denote the inverse of $e^{\hat{\mu}}_{\nu}$ with $e^{\nu}_{\hat{\rho}}$ (such that $e^{\hat{\mu}}_{\nu} e^{\nu}_{\hat{\rho}} = \delta^{\hat{\mu}}_{\hat{\rho}}$ and $e^{\nu}_{\hat{\mu}} e^{\hat{\mu}}_{\rho} = \delta^{\nu}_{\rho}$). We will need the spin connections, which can be found with[6]

$$\mathrm{d}e^{\hat{\mu}} =: -\omega^{\hat{\mu}}{}_{\hat{\nu}} \wedge e^{\hat{\nu}} \ , \tag{A.5}$$

implying (we denote with $'$ derivatives with respect to the argument of the respective functions)

$$\begin{aligned}
\mathrm{d}e^{\hat{0}} &= -\omega^{\hat{0}}{}_{\hat{j}} \wedge e^{\hat{j}} = 0 \ , \\
\mathrm{d}e^{\hat{1}} &= -\omega^{\hat{1}}{}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a'\,\mathrm{d}\eta \wedge \mathrm{d}w \ \ ; \\
\mathrm{d}e^{\hat{2}} &= -\omega^{\hat{2}}{}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a' f_\kappa\,\mathrm{d}\eta \wedge \mathrm{d}\theta + a f'_\kappa\,\mathrm{d}w \wedge \mathrm{d}\theta \ \ ; \\
\mathrm{d}e^{\hat{3}} &= -\omega^{\hat{3}}{}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a' f_\kappa \sin\theta\,\mathrm{d}\eta \wedge \mathrm{d}\varphi + a f'_\kappa \sin\theta\,\mathrm{d}w \wedge \mathrm{d}\varphi + a f_\kappa \cos\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi \ \ ;
\end{aligned} \tag{A.6}$$

From the previous, we can read (we can lower or raise hatted indices with the Minkowski metric, $\omega_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\rho}} \omega^{\hat{\rho}}{}_{\hat{\nu}}$ etc., and notice $\omega_{\hat{\mu}\hat{\nu}} = -\omega_{\hat{\nu}\hat{\mu}}$)

$$\begin{aligned}
\omega^{\hat{0}}{}_{\hat{1}} &= \frac{a'}{a}\,\mathrm{d}w = \omega^{\hat{1}}{}_{\hat{0}} \ , \\
\omega^{\hat{0}}{}_{\hat{2}} &= \frac{a'}{a} f_\kappa\,\mathrm{d}\theta = \omega^{\hat{2}}{}_{\hat{0}} \ , \\
\omega^{\hat{0}}{}_{\hat{3}} &= \frac{a'}{a} f_\kappa \sin\theta\,\mathrm{d}\varphi = \omega^{\hat{3}}{}_{\hat{0}} \ , \\
\omega^{\hat{1}}{}_{\hat{2}} &= -f'_\kappa\,\mathrm{d}\theta = -\omega^{\hat{2}}{}_{\hat{1}} \ , \\
\omega^{\hat{1}}{}_{\hat{3}} &= -f'_\kappa \sin\theta\,\mathrm{d}\varphi = -\omega^{\hat{3}}{}_{\hat{1}} \ , \\
\omega^{\hat{2}}{}_{\hat{3}} &= -\cos\theta\,\mathrm{d}\varphi = -\omega^{\hat{3}}{}_{\hat{2}} \ .
\end{aligned} \tag{A.7}$$

Once we have the spin connection ω, the Riemann tensor can be found exploiting[7]

$$R^{\hat{\mu}}{}_{\hat{\nu}} := \mathrm{d}\omega^{\hat{\mu}}{}_{\hat{\nu}} + [\omega^{\hat{\mu}}{}_{\hat{\sigma}} \wedge \omega^{\hat{\sigma}}{}_{\hat{\nu}}] = \frac{1}{2} e^{\hat{\mu}}_{\rho} e^{\sigma}_{\hat{\nu}} R^{\rho}{}_{\sigma\alpha\beta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \ \ ; \tag{A.9}$$

[6] Recall the formula for exterior derivative

$$\mathrm{d}(a_{\mu\ldots\sigma}\,\mathrm{d}x^\mu \wedge \ldots \wedge \mathrm{d}x^\sigma) = (\partial_\nu a_{\mu\ldots\sigma})\,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \wedge \ldots \wedge \mathrm{d}x^\sigma \ \ , \tag{A.3}$$

where the outer product of differentials is defined as

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2}(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu) \ . \tag{A.4}$$

[7] For reference, we show the Christoffel symbols formula in the tetrad formalism,

$$\Gamma^\sigma_{\mu\rho} = e^\sigma_{\hat{a}} \omega^{\hat{a}}{}_{\hat{b}\mu} e^{\hat{b}}_{\rho} + e^\sigma_{\hat{a}} \partial_\mu e^{\hat{a}}_{\rho} \ , \tag{A.8}$$

We need ($\mathcal{H} := a'/a$)

$$
\begin{aligned}
R^{\hat{0}}{}_{\hat{1}} &= \mathcal{H}' \,\mathrm{d}\eta \wedge \mathrm{d}\omega \;\; ; \\
R^{\hat{0}}{}_{\hat{2}} &= \mathcal{H}' f_\kappa \,\mathrm{d}\eta \wedge \mathrm{d}\theta \;\; , \\
R^{\hat{0}}{}_{\hat{3}} &= \mathcal{H}' f_\kappa \sin\theta \,\mathrm{d}\varphi \wedge \mathrm{d}\theta \;\; , \\
R^{\hat{1}}{}_{\hat{2}} &= \left(\mathcal{H}^2 f_\kappa - f_\kappa''\right) \mathrm{d}w \wedge \mathrm{d}\theta \;\; , \\
R^{\hat{1}}{}_{\hat{3}} &= \left(\mathcal{H}^2 f_\kappa - f_\kappa''\right) \sin\theta \,\mathrm{d}w \wedge \mathrm{d}\varphi \;\; , \\
R^{\hat{2}}{}_{\hat{3}} &= \left(\mathcal{H}^2 f_\kappa^2 - {f_\kappa'}^2 + 1\right) \sin\theta \,\mathrm{d}\theta \wedge \mathrm{d}\varphi \;\; ,
\end{aligned}
\tag{A.10}
$$

From this, one recovers (it should be obvious, but we remark that sum over repeated indices is not implied in the following equations)

$$
\begin{aligned}
e^{\hat{0}}_0 e^1_{\hat{1}} R^0{}_{1\mu\nu} = \frac{a}{a} R^0{}_{1\mu\nu} &\implies R^0{}_{101} = \mathcal{H}' \;, \\
e^{\hat{0}}_0 e^2_{\hat{2}} R^0{}_{2\mu\nu} = \frac{1}{f_\kappa} R^0{}_{2\mu\nu} &\implies R^0{}_{202} = \mathcal{H}' f_\kappa^2 \;, \\
e^{\hat{0}}_0 e^3_{\hat{3}} R^0{}_{3\mu\nu} = \frac{1}{f_\kappa \sin\theta} R^0{}_{3\mu\nu} &\implies R^0{}_{303} = \mathcal{H}' f_\kappa^2 \sin^2\theta \;, \\
e^{\hat{1}}_1 e^2_{\hat{2}} R^1{}_{2\mu\nu} = \frac{1}{f_\kappa} R^1{}_{2\mu\nu} &\implies R^1{}_{212} = \left(\mathcal{H}^2 f_\kappa - f_\kappa''\right) f_\kappa \;, \\
e^{\hat{1}}_1 e^3_{\hat{3}} R^1{}_{3\mu\nu} = \frac{1}{f_\kappa \sin\theta} R^1{}_{3\mu\nu} &\implies R^1{}_{313} = \left(\mathcal{H}^2 f_\kappa - f_\kappa''\right) f_\kappa \sin^2\theta \;, \\
e^{\hat{2}}_2 e^3_{\hat{3}} R^2{}_{3\mu\nu} = \frac{1}{\sin\theta} R^2{}_{3\mu\nu} &\implies R^2{}_{323} = \left(\mathcal{H}^2 f_\kappa^2 - {f_\kappa'}^2 + 1\right) \sin^2\theta \;.
\end{aligned}
\tag{A.11}
$$

Ricci tensor reads (use relations like $R^\mu{}_{\nu\mu\nu} = g^{\mu\rho} g_{\nu\sigma} R^\sigma{}_{\rho\nu\mu}$)

$$
\begin{aligned}
R_{00} &= R^i{}_{0i0} = -R^0{}_{101} - \frac{1}{f_\kappa^2} R^0{}_{202} - \frac{1}{f_\kappa^2 \sin^2\theta} R^0{}_{303} = -3\mathcal{H}' \;, \\
R_{11} &= \mathcal{H}' + 2\mathcal{H}^2 - 2\frac{f_\kappa''}{f_\kappa} \;, \\
R_{22} &= \mathcal{H}' f_\kappa^2 + 2\mathcal{H}^2 f_\kappa^2 - f_\kappa f_\kappa'' - {f_\kappa'}^2 + 1 \;, \\
R_{33} &= (\mathcal{H}' f_\kappa^2 + 2\mathcal{H}^2 f_\kappa^2 - f_\kappa f_\kappa'' - {f_\kappa'}^2 + 1) \sin^2\theta \;;
\end{aligned}
\tag{A.12}
$$

and the Ricci scalar

$$
a^2 R = a^2 g^{\mu\nu} R_{\mu\nu} = -6\mathcal{H}' - 6\mathcal{H}^2 + 4\frac{f_\kappa''}{f_\kappa} - \frac{2}{f_\kappa^2}(1 - {f_\kappa'}^2) = -6\mathcal{H}' - 6\mathcal{H}^2 - 6\kappa \;,
\tag{A.13}
$$

where on the last step we used the explicit form of f_κ for all possible κ.

When using Einstein equations, one should use the stress energy tensor in the new coordinates; in particular, notice that $u^\mu = (1/a, 0, 0, 0)$, hence $T_{\mu\nu} = a^2 \mathrm{diag}(\bar{\rho}, \bar{P}, \bar{P}, \bar{P})$. The first Einstein equation reads (we convert to comoving time t instead of conformal time η, so $\dot{} := \partial/\partial t$, $\mathcal{H} = aH$, $H = \dot{a}/a$)

$$
R_{00} - \frac{1}{2} a^2 R = 3a^2 H^2 + 3\kappa = 8\pi G a^2 \bar{\rho} + \Lambda a^2 \implies H^2 = \frac{8\pi G}{3}\bar{\rho} + \frac{\Lambda}{3} - \frac{\kappa}{a^2} \;.
\tag{A.14}
$$

The 11 Einstein equation instead reads

$$
R_{11} + \frac{1}{2} a^2 R = a^2 \left(\frac{-2\ddot{a}}{a} - H^2\right) - \kappa = 8\pi G a^2 \bar{P} - \Lambda a^2 \;,
\tag{A.15}
$$

ending with the second Friedmann equation (using the first to substitute for H^2)

$$
\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) + \frac{\Lambda}{3} \;.
\tag{A.16}
$$