Cosmology 2 TA: General relativity primer

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1 General covariance

General Relativity (GR) is a generically covariant theory, meaning that its equations are expressed with objects which are covariant with respect to generic coordinate transformation. In other words, GR is independent on which coordinate system (or reference frame) one uses. This seems like a trivial statement, but notice that, for example, the famous second Newton law

$$\vec{F} = m\vec{a} \ , \tag{1.1}$$

is not covariant under general coordinate changes. Going to an accelerated reference frame, you end up with spurious apparent forces an accelerated observer has to introduce to explain what he sees. Eq. (1.1) is however covariant under rotations. This is not surprising, since Eq. (1.1) is expressed using vectors, which have precise behaviour (a covariant behaviour) with respect of rotations.

What then one needs in GR is to express quantities with respect to tensors. A tensor $T^{\gamma\delta\dots}_{\alpha\beta\dots}(y)$ of rank (p,q), when undercomes a coordinate change $y \to x$, transforms as

$$T^{\mu_1\dots\mu_p}_{\nu_1\dots\nu_q}(x) = \left(\frac{\partial y^{\alpha_1}}{\partial x^{\nu_1}} \cdots \frac{\partial y^{\alpha_p}}{\partial x^{\nu_p}}\right)\left(\frac{\partial x^{\mu_1}}{\partial y^{\beta_1}} \cdots \frac{\partial x^{\mu_p}}{\partial y^{\beta_p}}\right) T^{\beta_1\dots\beta_p}_{\alpha_1\dots\alpha_q}(y) \ ; \tag{1.2}$$

scalar quantities (which “have no indices”) do not change under coordinate transformation.

For an easy example, you know that, in the coordinates where FRW line element reads

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2\,\mathrm{d}y^i\,\mathrm{d}y^i \ \ , \tag{1.3}$$

(where $t = y^0$) a comoving observer 4-velocity would read

$$u^\mu(y) = (1,0,0,0) \ ; \tag{1.4}$$

when one wants to use conformal time, $\mathrm{d}\eta = \mathrm{d}t\,/a$, one has $(\eta = x^0, x^i = y^i)$

$$u^0(x) = \frac{\partial x^0}{\partial y^\mu} u^\mu(y) = \frac{\partial \eta}{\partial t} u^0(x) = 1/a \ , \tag{1.5}$$

which is what one expects. Notice also that the metric coefficients transform as

$$g_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(y) \ . \tag{1.6}$$

Indeed, $g_{00}(x) = a^2 g_{00}(y) = a^2$.

2 Making quantities covariant

Suppose we have a scalar function $f(x)$. Is $\frac{\partial f}{\partial x^\mu}$ covariant? The answer is trivially yes, due to the chain rule. In particular, you obtain a $(0,1)$ tensor from the scalar f, which is a $(0,0)$ tensor. But it is also easy to see that the derivative of a vector transforms as

$$\frac{\partial u^\mu(x)}{\partial x^\nu} = \frac{\partial y^\alpha}{\partial x^\nu}\frac{\partial}{\partial y^\alpha}\left(\frac{\partial x^\mu}{\partial y^\beta}u^\beta(y)\right) = \frac{\partial y^\alpha}{\partial x^\nu}\frac{\partial x^\mu}{\partial y^\beta}\frac{\partial u^\mu}{\partial y^\alpha} + \left[\frac{\partial y^\alpha}{\partial x^\nu}\frac{\partial^2 x^\mu}{\partial y^\alpha y^\beta}\right] \ ; \tag{2.1}$$

the term in square brackets is what "breaks" covariance. It turns out that proper way to define a covariant derivative is

$$\mathrm{D}_\mu u^\nu = \partial_\mu u^\nu + \Gamma^\nu_{\mu\sigma} u^\sigma \ , \tag{2.2}$$

where $\Gamma^\mu_{\nu\rho}$ is the Christoffel symbol (notice the analogy with gauge theories in QFT). The covariant derivative changes a (p, q) tensor to a $(p, q+1)$ tensor.

Another issue comes in integration; in fact, under generic coordinate change, the volume element changes as

$$\mathrm{d}^4 x = \det\left(\frac{\partial x^\mu}{\partial y^\nu}\right) \mathrm{d}^4 y \ ; \tag{2.3}$$

not only it is not covariant, but we would expect the volume element to be invariant (if I integrate a scalar, I want to end up with another scalar). It is then a straightforward consequence of Eq. (1.6) to show that an invariant volume element definition is

$$\mathrm{d}^4 x \sqrt{-\det(g)} \ ; \tag{2.4}$$

moreover, with this definition, the volume element in special relativity is just $\mathrm{d}^4 x$, since $-\det(g) = 1$ for Minkowski (a Lorentz transformation does not change this result, since $\det\Lambda = 1$, where Λ is a Lorentz transformation matrix). It makes sense that the metric tells how volume elements are constructed; to convince yourself that this is the correct way to define the invariant volume element, consider the $3D$ flat metric in spherical coordinates

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\varphi^2) \ ; \tag{2.5}$$

then Eq. (2.4) (which for this specific case becomes $\mathrm{d}^3 x = \sqrt{\det g}\, \mathrm{d}^3 y$) tells us that

$$\mathrm{d}^3 x = r^2 \sin\theta\, \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\varphi \ , \tag{2.6}$$

as expected.

This suggests the prescriptions to pass from Special Relativity (SR) to GR: substitute

$$\eta_{\mu\nu} \to g_{\mu\nu} \ , \ \partial_\mu \to D_\mu \ , \ \mathrm{d}^4 x \to \sqrt{-\det g}\, \mathrm{d}^4 x \ . \tag{2.7}$$

3 Scalar field in GR

Using all the prescriptions in Eq. (2.7) for the action of a scalar field, we have (recall that covariant derivatives on scalars are equivalent to usual partial derivatives)

$$S_{\text{matter}} = \int \mathrm{d}^4 x \sqrt{-\det g}\left(\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right) =: \int \mathrm{d}^4 x \sqrt{-\det g}\mathcal{L} \ ; \tag{3.1}$$

to the previous, we should add the part of the action which will give rise to the LHS of the Einstein equations. This part is called Einstein-Hilbert action, and reads

$$S_{\text{EH}} = \frac{1}{16\pi G}\int \mathrm{d}^4 x \sqrt{-\det g} R \ , \tag{3.2}$$

where R is the Ricci scalar. Then $S = S_{\text{EH}} + S_{\text{matter}}$. The equation of motion for ϕ follows from

$$\frac{\delta S}{\delta\phi} = 0 \implies \frac{\partial\sqrt{-\det g}\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\sqrt{-\det g}\mathcal{L}}{\partial\partial_\mu\phi} \ , \tag{3.3}$$

which yields

$$\frac{1}{\sqrt{-\det g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-\det g} g^{\mu\nu}\partial_\nu\phi\right) + \frac{\partial V}{\partial\phi} = 0 \ . \tag{3.4}$$

Specialized to FRW, we obtain ($\phi = \phi(t)$)

$$\frac{1}{a^3}\partial_t(a^3\partial_t\phi) + \frac{\partial V}{\partial\phi} = 0 \implies \partial_t^2\phi + 3H\phi + \frac{\partial V}{\partial\phi} = 0 \ ; \tag{3.5}$$

notice the $3H\phi$ term, which is a friction term coming from the expansion of the universe.

Einstein equations are obtained via

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \implies \frac{1}{16\pi G}\left(\frac{\partial\sqrt{-\det g}R}{\partial g^{\mu\nu}} - \left[\partial_\sigma \frac{\partial\sqrt{-\det g}R}{\partial\partial_\sigma g^{\mu\nu}}\right]\right) + \frac{\partial\sqrt{-\det g}\mathcal{L}}{\partial g^{\mu\nu}} \ . \tag{3.6}$$

To compute such a functional derivative, one needs

$$\frac{\partial\sqrt{-\det g}}{\partial g^{\mu\nu}} = -\frac{1}{2\sqrt{-\det g}}\frac{\partial \det(g)}{\partial g^{\mu\nu}} \ ; \tag{3.7}$$

notice that we can write

$$\det g = \mathrm{e}^{\ln\det g} = \mathrm{e}^{\mathrm{Tr}\ln g} \ , \tag{3.8}$$

implying

$$\frac{\partial \det g}{\partial g^{\mu\nu}} = \det g \,\mathrm{Tr}\left(g^{-1}\frac{\partial g}{\partial g^{\mu\nu}}\right) = \det(g) g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial g^{\mu\nu}} \ ; \tag{3.9}$$

notice

$$\frac{\partial g_{\rho\sigma}g^{\rho\sigma}}{\partial g^{\mu\nu}} = \frac{\partial\delta^\rho_\rho}{\partial g^{\mu\nu}} = 0 \implies g^{\rho\sigma}\frac{\partial g_{\rho\sigma}}{\partial g^{\mu\nu}} = -g_{\rho\sigma}\frac{\partial g^{\rho\sigma}}{\partial g^{\mu\nu}} = -g_{\mu\nu} \ , \tag{3.10}$$

ending up with

$$\frac{\partial\sqrt{-\det g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-\det g}}{2}g_{\mu\nu} \ . \tag{3.11}$$

We thus have

$$\frac{1}{16\pi G}\frac{\partial\sqrt{-\det g}g^{\rho\sigma}R_{\rho\sigma}}{\partial g^{\mu\nu}} = \frac{\sqrt{-\det g}}{16\pi G}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) + \left[\frac{\sqrt{-\det g}}{16\pi G}g^{\rho\sigma}\frac{\partial R_{\rho\sigma}}{\partial g^{\mu\nu}}\right] \ ; \tag{3.12}$$

one recognizes the LHS of Einstein equations on the round brackets, whereas the term on square brackets is a boundary term (which cancels with the term in square brackets in Eq. (3.6)); this takes a little too long to show, hence it is left as an exercise. Then, one has finally, from Eq. (3.6)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -8\pi G\left(\frac{2}{\sqrt{-\det g}}\frac{\partial\sqrt{-\det g}\mathcal{L}}{\partial g^{\mu\nu}}\right) \ . \tag{3.13}$$

the previous is exactly the Einstein equation (using the sign conventions used in class), granted one identifies (from now on, $g := \det g$ as it is customary)

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{matter}}}{\delta g^{\mu\nu}} \ . \tag{3.14}$$

The energy-momentum tensor reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi - V(\phi)\right) \ . \tag{3.15}$$