Cosmology 2 TA: Power-law inflation

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1 The model of power-law inflation

You saw in class that one can study the fluctuations evolution in an inflationary scenario by looking at the Sasaki-Mukhanov equation

$$\frac{\mathrm{d}^2 R_q}{\mathrm{d}\eta^2} + \frac{2}{z}\frac{\mathrm{d}z}{\mathrm{d}\eta}\frac{\mathrm{d}R_q}{\mathrm{d}\eta} + q^2 R_q = 0 \ , \ R := \Psi + \frac{a\varphi}{z} \ , \ z := \frac{a\dot{\bar{\phi}}}{H} \ . \tag{1.1}$$

where Ψ is the gravitational potential (in the Newtonian gauge) and η is the conformal time. We denote the scalar field by

$$\phi = \bar{\phi} + \varphi \ , \tag{1.2}$$

where $\bar{\phi}$ is the background value and φ the fluctuation around the background. R_q are the Fourier components of the gauge invariant quantity R. The use of R over the field Ψ, φ comes from the fact that R fluctuations are conserved outside the horizon (plus, it is a gauge invariant quantity). This allows us to treat perturbations coming from inflation, whose horizon re-entry corresponds to well understood epochs (like from BBN onwards).

We can discuss Eq. (1.1) analytically for an exponential potential of the form

$$V(\phi) = V_0 \mathrm{e}^{-\lambda\phi} \ . \tag{1.3}$$

The inflationary model with this kind of potential is called power-law inflation, for a reason we will soon see. The energy of the background scalar field will read

$$\rho = \frac{1}{2}\dot{\bar{\phi}}^2 + V_0 \mathrm{e}^{-\lambda\bar{\phi}} \ ; \tag{1.4}$$

from its equation of motion

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \frac{\partial V(\bar{\phi})}{\partial\phi} = 0 \ , \tag{1.5}$$

and the first Friedmann equation

$$H^2 = \frac{8\pi G}{3}\left(\frac{1}{2}\dot{\bar{\phi}}^2 + V_0 \mathrm{e}^{-\lambda\bar{\phi}}\right) \ . \tag{1.6}$$

Deriving Friedmann, and then using Eq. (1.5)

$$2H\dot{H} = \frac{8\pi G}{3}(\dot{\bar{\phi}}\ddot{\bar{\phi}} + V'(\bar{\phi})\dot{\bar{\phi}}) = \frac{8\pi G}{3}(-3H\dot{\bar{\phi}}^2) \implies \dot{H} = -4\pi G\dot{\bar{\phi}}^2 \ . \tag{1.7}$$

This model is particular, in which there is an exact solution to Eqs. (1.6)(1.5), which is (just substitute and see)

$$\bar{\phi} = \frac{1}{\lambda}\ln\frac{8\pi G V_0 \epsilon^2 t^2}{3-\epsilon} \ , \ H = \frac{1}{t\epsilon} \ , \ \epsilon = \frac{\lambda^2}{16\pi G} \ ; \tag{1.8}$$

we can verify that ϵ as defined here is indeed the slow-roll parameter for this model

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{4\pi G V'^2}{9H^4} = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \frac{\lambda^2}{16\pi G} \ ; \tag{1.9}$$

notice that, since ϵ is constant,

$$H = \frac{1}{\epsilon t} \implies a(t) = a_0\left(\frac{t}{t_0}\right)^{1/\epsilon} \; ; \tag{1.10}$$

this behaviour of the scale factor is why this model is called power-law inflation. Notice the condition for accelerated expansion

$$\ddot{a} \propto \frac{1}{\epsilon}\left(\frac{1}{\epsilon} - 1\right)t^{1/\epsilon - 2} \implies \ddot{a} > 0 \text{ if } \frac{1-\epsilon}{\epsilon} > 0 \; , \tag{1.11}$$

hence $\epsilon < 1$. Thanks to Eq. (1.10), we can easily express the δ slow-roll parameter

$$\delta := \frac{1}{2}\frac{\ddot{H}}{H\dot{H}} = -\epsilon \; . \tag{1.12}$$

We can then express the conformal time as

$$\eta \propto \int_{-\infty}^{t} \frac{\mathrm{d}t'}{a} = -a_0 t_0^{1/\epsilon}\frac{\epsilon}{1-\epsilon}t^{-(1-\epsilon)/\epsilon} \sim -\frac{1}{aH} \; , \tag{1.13}$$

and for simplicity, we set $a_0 = 1$ so that (I keep t_0 to keep track of dimensions)

$$\eta = -t_0^{1/\epsilon}\frac{\epsilon}{1-\epsilon}t^{-(1-\epsilon)/\epsilon} \implies t = t_0\left(-\frac{1-\epsilon}{\epsilon t_0}\eta\right)^{-\epsilon/(1-\epsilon)} \; ; \tag{1.14}$$

with Eq. (1.8), we can write

$$\dot{\bar{\phi}} = \frac{2}{\lambda t} \implies \frac{2}{z}\frac{\mathrm{d}z}{\mathrm{d}\eta} = -\frac{2}{(1-\epsilon)\eta} \; . \tag{1.15}$$

We can finally write the Sasaki-Mukhanov equation in the form (with $s := -\eta q$)

$$\frac{\mathrm{d}^2 R_q}{\mathrm{d}s^2} - \frac{2\nu - 1}{s}\frac{\mathrm{d}R_q}{\mathrm{d}s} + R_q = 0 \; , \; \nu = \frac{3-\epsilon}{2(1-\epsilon)} \tag{1.16}$$

This is a Bessel equation, whose solution we can write as a linear combination of first and second kind Hankel functions[1] ,

$$R_q(s) = s^\nu(a_1 H_\nu^{(1)}(s) + a_2 H_\nu^{(2)}(s)) \; , \tag{1.19}$$

where $H_\nu^{(1)} = H_\nu^{(2)*}$. We need the following asymptotic behavior of Hankel functions

$$\begin{cases} H_\nu^{(1)}(s) \to -\sqrt{\frac{2}{\pi}}\mathrm{e}^{-\frac{1}{4}\mathrm{i}\pi(2\nu-3)}\frac{\mathrm{e}^{\mathrm{i}s}}{\sqrt{s}} \; , \; s \gg 1 \text{ inside horizon} \; , \\ H_\nu^{(1)}(s) \to -\mathrm{i}\frac{2^\nu\Gamma(\nu)}{\pi}s^{-\nu} \; , \; s \ll 1 \text{ outside horizon} \; . \end{cases} \tag{1.20}$$

We can give the initial conditions deep inside the horizon, $-q\eta = s \gg 1$, where we can set

$$\varphi_q \sim \frac{\mathrm{e}^{-\mathrm{i}q\eta}}{a} \; , \; \Psi_q \sim -\mathrm{i}\frac{4\pi G\dot{\bar{\phi}}}{sH^2}a\varphi_q \sim \mathcal{O}\left(\frac{1}{s}\right) \; , \tag{1.21}$$

in particular, we can neglect Ψ_q in the expression of R_q; this leads us to

$$R_q \underset{s\gg 1}{\to} -s^\nu\sqrt{\frac{2}{s\pi}}(a_1\mathrm{e}^{-\mathrm{i}\pi(2\nu-3)/4}\mathrm{e}^{\mathrm{i}s} + a_2\mathrm{e}^{\mathrm{i}\pi(2\nu-3)/4}\mathrm{e}^{-\mathrm{i}s}) \sim \frac{H\varphi_q}{\dot{\bar{\phi}}} \sim \frac{H}{a\dot{\bar{\phi}}}\mathrm{e}^{\mathrm{i}s} \; , \tag{1.22}$$

this implies $a_2 = 0$ and

$$a_1 = -\frac{H}{as^\nu\dot{\bar{\phi}}}\sqrt{\frac{\pi s}{2}}\mathrm{e}^{\mathrm{i}\pi(2\nu-3)/4} \; ; \tag{1.23}$$

[1] A Bessel equation of the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - \frac{2\nu-1}{x}\frac{\mathrm{d}y}{\mathrm{d}x} + \left(\beta^2\gamma^2 x^{2\gamma-2} + \frac{\nu^2 - n^2\gamma^2}{x^2}\right)y = 0 \; , \tag{1.17}$$

has solutions

$$y = x^\nu(AH_n^{(1)}(\beta x^\gamma) + BH_n^{(2)}(\beta x^\gamma)) \; , \tag{1.18}$$

if $n \neq \mathbb{N}$. In our case, $n^2 = \nu^2$, $\beta = \gamma = 1$.

we end up with, noticing (we use Eq. (1.8),(1.14))

$$\frac{H}{a\dot{\phi}} = \frac{\lambda}{2\epsilon}\left(\frac{1-\epsilon}{\epsilon}\frac{(-\eta)}{t_0}\right)^{1/(1-\epsilon)} \tag{1.24}$$

with

$$R_q = -\frac{\lambda}{2\epsilon}\sqrt{\frac{\pi q t_0}{2}}\left(\frac{1-\epsilon}{\epsilon}\right)^{1/(1-\epsilon)}\left(\frac{-\eta}{t_0}\right)^{\nu} \mathrm{e}^{\mathrm{i}\pi(2\nu-3)/4} H_\nu^{(1)}(s) \ . \tag{1.25}$$

Outside the horizon, Eq. (1.25) reads

$$R_q \to \frac{2^{\nu-3/2}\Gamma(\nu)\lambda t_0^{1/2-\nu}}{\epsilon\sqrt{\pi}}\left(\frac{1-\epsilon}{\epsilon}\right)^{1/(1-\epsilon)} \mathrm{e}^{\mathrm{i}\pi(2\nu-3)/4} q^{1/2-\nu} \ ; \tag{1.26}$$

notice the time independence (as expected) and the q behaviour, which is typical of slow-roll inflation and not limited to the power-law inflation.

We can finally relate this quantity to physical observables, by looking at (for super-horizon modes)

$$\langle R^2 \rangle = \int \frac{\mathrm{d}^3 q}{(2\pi)^3 2q}|R_q^2| = \frac{t_0^{1-2\nu}}{2\pi^2}\int \mathrm{d}\ln q\, \frac{2^{2\nu-3}\Gamma^2(\nu)\lambda^2}{\epsilon^2\pi}\left(\frac{1-\epsilon}{\epsilon}\right)^{2/(1-\epsilon)} q^{3-2\nu} =: \int \mathrm{d}\ln q\, \Delta_R^2(q) \ ; \tag{1.27}$$

2 Connecting with observables

Δ_R^2 is connected to the density fluctuation power spectrum, and it is an observable (for example, by looking at correlations in the CMB spectrum). It reads

$$\Delta_R^2 = \frac{2^{2\nu-4}\Gamma^2(\nu)\lambda^2 t_0^{1-2\nu}}{\epsilon^2\pi^3}\left(\frac{1-\epsilon}{\epsilon}\right)^{2/(1-\epsilon)} q^{3-2\nu} =: A_\mathrm{s}\left(\frac{q}{q_*}\right)^{n_\mathrm{s}-1} , \tag{2.1}$$

where on the last equality we used the usual parametrization of Δ_R^2, with the pivot scale $q_* = 0.05\,\mathrm{Mpc}^{-1}$. From CMB, we measured

$$A_\mathrm{s} \simeq 2\times 10^{-9} \ , \ n_\mathrm{s} \simeq 0.96 \ . \tag{2.2}$$

For the power law model, this implies

$$n_\mathrm{s} = 4 - 2\nu = \frac{1-3\epsilon}{1-\epsilon} \simeq 1 - 3\epsilon \implies \epsilon = \frac{\lambda^2}{16\pi G} \simeq 0.015 \ , \tag{2.3}$$

which is compatible with all our slow-roll assumptions, necessary for inflation to work. To connect with the results one obtains with other inflationary potentials (and to remove the annoying t_0), we can express A_s for this model by using $H(t_q)$, where t_q is the time the fluctuations characterized by q re-enters the horizon (this is where a slight q dependence enters in other potentials, here it is not really there). We have

$$\frac{q}{a(t_q)} = H(t_q) \implies t_q = (\epsilon q)^{\epsilon/(1-\epsilon)} t_0^{1/(1-\epsilon)} \implies H(t_q) = \frac{1}{\epsilon t_0^{1/(1-\epsilon)}(\epsilon q)^{\epsilon/(1-\epsilon)}} \ . \tag{2.4}$$

We can then write (use $\lambda^2 = 16\pi G\epsilon$)

$$A_\mathrm{s} = \Delta_R^2(q_*) = \frac{H^2(t_{q_*})}{\pi^2} 2^{2\nu}\Gamma^2(\nu) G \epsilon^{2\epsilon/(1-\epsilon)}\left(\frac{1-\epsilon}{\epsilon}\right)^{2/(1-\epsilon)} \tag{2.5}$$

(notice $3 - 2\nu = -2\epsilon/(1-\epsilon)$ and $1 - 2\nu = -2/(1-\epsilon)$).

We see that ($\nu \simeq 3/2$, $\Gamma(\nu) \simeq \sqrt{\pi}/2$)

$$H^2(t_{q_*}) \simeq \frac{A_\mathrm{s}\pi}{2G}\epsilon^{(1+\epsilon)/(\epsilon-1)}\left(\frac{1-\epsilon}{\epsilon}\right)^{-2/(1-\epsilon)} \implies H(t_{q_*}) \simeq 1\times 10^{14}\,\mathrm{GeV} \ , \tag{2.6}$$

which is consistent with expectations regarding the inflation scale.

Is the exponential potential then a good inflation model? No, because inflation needs to end, and in this scenario it does not. However, all our derivations rely on the fact that the potential is exponential just at around horizon crossing, not everywhere. With this limitation, is this scenario allowed by observations? Unfortunately no. You will (probably) see in the lecture the fluctuation contribution from gravitational waves (tensor contribution). The tensor to scalar ratio

$$r_{\mathrm{T}} = \frac{\Delta_{\mathrm{T}}^2}{\Delta_R^2} \tag{2.7}$$

is measurable, and under very generic conditions (exponential potential included) the relation

$$r = 16\epsilon \ , \tag{2.8}$$

holds. This can be shown in the following. If we choose a gauge where $\Psi = 0$, then R, being gauge invariant, is unaffected, and we can write

$$R = \frac{H}{\dot{\phi}}\varphi \implies \Delta_R^2 = \frac{\lambda^2}{4\epsilon^2}\Delta_\varphi^2 = \frac{4\pi G}{\epsilon}\Delta_\varphi^2 \ ; \tag{2.9}$$

the relevance of this is that from the gravitational waves action, (from $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(\delta_{ij} + h_{ij})\,\mathrm{d}x^i\,\mathrm{d}x^j$)

$$S = \frac{1}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}R \ , \tag{2.10}$$

one obtains the same equations for a scalar, once one normalizes $h_{ij} = \sqrt{32\pi G}f_{ij}$. Then

$$\Delta_{\mathrm{T}}^2 = 2 \times 32\pi G\Delta_\varphi^2 \ , \tag{2.11}$$

where the factor 2 comes from the 2 gravitational waves polarizations. Then one recovers Eq. (2.8).

For our model, $r \simeq 0.25$, and this exceeds the experimental bound given by CMB, hence power-law inflation is ruled out.

A Units

Quantization of fields is done via

$$\varphi = \int \frac{\mathrm{d}\vec{q}}{(2\pi)^{3/2}\sqrt{2q}}(\varphi_q a_{\vec{q}}\mathrm{e}^{\mathrm{i}\vec{q}x} + \varphi_{\vec{q}}^* a_q^\dagger \mathrm{e}^{-\mathrm{i}\vec{q}x}\ )\Big|_{q^0=\omega_q} \tag{A.1}$$

a_q, $a_q^\dagger$ obey the commutation relation

$$[a_q, a_{q'}^\dagger]_{q^0=q'^0} = \delta(\vec{q} - \vec{q}') \ , \tag{A.2}$$

hence a_q has mass dimensions $-3/2$, so φ_q is dimensionless (since φ has mass dimension 1). For the same reason, R_q has mass dimensions -1.

B Horizon problem

To solve the horizon and flatness problem, we need (exploiting the previous)

$$\frac{a(t_2)}{a(t_1)} = \left(\frac{t_2}{t_1}\right)^{1/\epsilon} = \exp\left(\frac{\lambda}{2\epsilon}(\phi_2 - \phi_1)\right) \sim \mathrm{e}^{60} \ . \tag{B.1}$$

Notice that λ has dimensions of inverse mass. By setting $\lambda = \alpha/M_{\mathrm{inf}}$ as the mass scale, we can see that the field excursion $\Delta\phi = \phi_2 - \phi_1$ obeys

$$\frac{\Delta\phi}{M_{\mathrm{inf}}} \simeq \frac{120\epsilon}{\alpha} \ ; \tag{B.2}$$

we saw that $\epsilon \simeq 0.02$, hence the field excursion during inflation is of order the mass scale M_{inf}, which for this model is . The excursion is not really a scale.