Cosmology 2 TA: Ultra-Light Dark Matter

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1 An ultralight axion

Consider a scalar field as a massless Goldstone boson, associated with spontaneous symmetry breaking at a scale f of some beyond standard model symmetry; this breaking would happen in the early universe. For concreteness, we can consider a complex scalar field φ, and the symmetry to be $U(1)$. At the symmetry breaking scale f,

$$\varphi = \chi e^{i\phi/f} \ , \ \langle \chi \rangle = \frac{f}{\sqrt{2}} \ . \tag{1.1}$$

The angular field ϕ is the scalar field we are interested about. Non-perturbative effects can generate a potential for this ϕ, but, given the identification

$$\frac{\phi}{f} \sim \text{angle} \ , \tag{1.2}$$

we can only allow a potential which is periodic of 2π; consider one of the form

$$V(\phi) = \Lambda^4 (1 - \cos(\phi/f)) \ , \tag{1.3}$$

where Λ is the scale where non-perturbative effects kick in. For $\phi < f$, we can expand

$$V(\phi) \simeq \frac{1}{2} m^2 \phi^2 \ , \ m^2 = \frac{\Lambda^4}{f^2} \ ; \tag{1.4}$$

since this potential comes from non-perturbative effects, we expect m to be very small (and all next corrections to be completely negligible). All other possible couplings to standard model fields are similarly suppressed.

The construction we sketched here appears in many BSM theories, most notably the QCD axion. This motivates to look into the cosmology of ultralight fields. For ultralight, we mean masses 10^{-22} eV$\lesssim m \lesssim 10^{-3}$ eV. Can such a low mass particle be dark matter? Notice that the occupation number of these particles, if they were to be dark matter, must be enormous

$$\mathcal{N} = \frac{\Delta N}{d^3x \, d^3k} \simeq \frac{n}{k^3} \simeq \frac{\rho_{\rm dm}}{m(mv)^3} \simeq 10^{87} \left(\frac{\rho_{\rm dm}}{0.4 \, \text{GeV cm}^{-3}} \right) \left(\frac{10^{-20} \, \text{eV}}{m} \right)^4 \left(\frac{200 \, \text{km s}^{-1}}{v} \right)^3 \ ; \tag{1.5}$$

due to Pauli exclusion principle, we cannot consider an ultralight fermion as a dark matter candidate (Tremaine-Gunn bound). Can such a particle be a good dark matter candidate? Can we produce enough of it, and if so, does it behave like cold dark matter?

In the next section, we will see a mechanism, a non-thermal (differently from freeze-out) mechanism for dark matter production, relevant for what it is called UltraLight Dark Matter (ULDM). We will not have any particular model in mind, we just consider a generic free massive scalar field minimally coupled to gravity.

2 The misalignment mechanism

A massive scalar field ϕ in an expanding universe obeys

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \ ; \tag{2.1}$$

if $a \propto t^p$, which holds in radiation or matter domination for example, we have $H = p/t$, and we can recast the previous as a Bessel equation, with $s := mt$, $' := \mathrm{d}/\mathrm{d}s$,

$$\phi'' - \frac{2\nu - 1}{s}\phi' + \phi = 0 \ , \ \nu = \frac{1-3p}{2} \ , \tag{2.2}$$

As we know, we can write this as a linear combination of first and second kind Hankel functions,

$$\phi \propto s^\nu (a_1 H_\nu^{(1)}(s) + a_2 H_\nu^{(2)}(s)) \propto a^{-3/2} t^{1/2} (a_1 H_\nu^{(1)}(s) + a_2 H_\nu^{(2)}(s)) \ , \tag{2.3}$$

where we used $t^\nu \propto a^{-3/2} t^{1/2}$. Recall the asymptotic behavior of Hankel functions

$$\begin{cases} H_\nu^{(1)}(s) \sim \dfrac{\mathrm{e}^{\mathrm{i}s}}{\sqrt{s}} \ , \ s \gg 1 \ , \\ H_\nu^{(1)}(s) \sim s^{-\nu} \ , \ s \ll 1 \ . \end{cases} \tag{2.4}$$

notice that, since $H \propto 1/t$, the regime $s \ll 1$ (very early times) is the regime $H \gg m$, where the damping coming from the expansion of the universe dominates and the field is stuck at its initial value, whereas for $s \gg 1$, harmonic-like oscillations dominate, as the asymptotic behavior of $H_\nu^{(1)}(s)$ also shows. In particular, at very early times, we can write

$$\dot{\phi}_\mathrm{i} = 0 \ , \ \phi_\mathrm{i} = f\theta_\mathrm{i} \ , \tag{2.5}$$

where the initial condition was inspired from the interpretation (see e.g. Eq. (1.3)) that ϕ/f is an angle. In particular, one can expect to set $\theta_\mathrm{i} = \mathcal{O}(1)$ in the stuck initial conditions. In the regime $s \ll 1$, this field behaves as dark energy. In the regime $s \gg 1$, we have

$$\phi \propto a^{-3/2} \mathrm{e}^{imt} \ , \tag{2.6}$$

in particular,

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2} m^2 \phi^2 \propto a^{-3} \ , \tag{2.7}$$

which is the expected behavior of a dark matter candidate. We can say that the time of the transition between the dark energy and the dark matter regime happens for $s = 1$ (it is an approximation, in the following we are interested only on estimates). Assuming this transition happens during radiation domination, where $a = a_0 (t/t_0)^{1/2}$, we can estimate

$$mt = m\left(\frac{a_\mathrm{osc}}{a_0}\right)^2 t_0 = 1 \implies a^2 = \frac{1 \times 10^{-20}\,\mathrm{eV}}{m} 1 \times 10^{-20}\,\mathrm{eV}^{-1} \times 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}} \ , \tag{2.8}$$

where we approximated $t_0 \sim 1/H_0$. This yields the approximate scale factor a_osc at which the field starts to oscillate to be

$$a \sim 10^{-6} \frac{1 \times 10^{-20}\,\mathrm{eV}}{m} \ , \tag{2.9}$$

which is after BBN, deep into the radiation dominated era. Approximating

$$\rho(a) \sim \rho(a_\mathrm{osc}) \left(\frac{a_\mathrm{osc}}{a}\right)^3 \ , \tag{2.10}$$

where $\rho(a_\mathrm{osc})$ is the initial, stuck density value before oscillation starts, which we write as

$$\rho(a_\mathrm{osc}) = \frac{1}{2} m^2 f^2 \theta_\mathrm{i}^2 \ , \tag{2.11}$$

we can estimate the critical parameter

$$\Omega = \frac{3H_0^2}{8\pi G}\rho(a=1) \sim \frac{8\pi}{3}\left(\frac{f\theta_\mathrm{i}}{M_\mathrm{pl}}\right)^2 \left(\frac{m}{1 \times 10^{-20}\,\mathrm{eV}}\right)^{1/2} 10^{-18} \frac{1}{H_0^2} \ . \tag{2.12}$$

Assuming $\theta_\mathrm{i} = \mathcal{O}(1)$, we end up with the estimate

$$\Omega \simeq 0.1 \left(\frac{f}{1 \times 10^{16}\,\mathrm{GeV}}\right)^2 \left(\frac{m}{1 \times 10^{-20}\,\mathrm{eV}}\right)^{1/2} \ . \tag{2.13}$$

This is the misalignment mechanism for dark matter production, called like this because it relies on the misalignment of the initial value θ_i from zero to determine the dark matter abundance. We remark that this mechanism is not thermal.

3 Wave dark matter in galaxies: the Schrödinger-Poisson equations

Most of the interesting features of ULDM comes from (cosmological) small scales, i.e. at the galactic scale. Those are the scales where this DM candidate clearly differentiate from the standard collisionless DM, and those are the scales where we can hope to detect (or more realistically, set bounds on) ULDM. This is due to the associated De-Broglie wavelength of ULDM, which is of astrophysics scale

$$\lambda \sim 6\,\mathrm{pc}\left(\frac{1\times 10^{-20}\,\mathrm{eV}}{m}\right)\left(\frac{2\times 10^{2}\,\mathrm{km\,s^{-1}}}{v}\right) . \tag{3.1}$$

The relevant action reads

$$S = \int \mathrm{d}^4x\,\sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right) ; \tag{3.2}$$

the metric we assume is the first order perturbed flat Robertson-Walker metric in Newtonian gauge, neglecting tensor and residual vector modes. It reads

$$\mathrm{d}s^2 = -(1+2\Phi)\,\mathrm{d}t^2 + a^2(t)(1-2\Phi)\,\mathrm{d}\vec{x}^2 \ ; \tag{3.3}$$

(neglecting the anisotropic part of the stress-energy tensor allows us to equate the two scalar field perturbations).

By varying the action with respect to the field, one can obtain the equation of motion

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi) = m^2\phi . \tag{3.4}$$

To first order in Φ and neglecting its time variation (weak field limit), recalling $\sqrt{-g} = (a^6(1+2\Phi)(1-2\Phi)^3)^{1/2} \simeq a^3(1-2\Phi)$, we have

$$-3H(1-2\Phi)\partial_0\phi - (1-2\Phi)\partial_0^2\phi + (1+2\Phi)\frac{\nabla^2\phi}{a} = m^2\phi , \tag{3.5}$$

where $H = \dot{a}/a$.

We are studying a Bose system with very high occupation number in phase space, so we can approximate it using a single wave function describing a classical field, since all interactions besides gravity can be neglected. Quantum corrections are expected to be completely negligible, since

$$\delta\hat{\phi} \sim \frac{1}{\sqrt{\mathcal{N}}} \tag{3.6}$$

(in other words, we are assuming the validity of mean field theory, where the mean values of operators are large compared to the root variance around those values due to quantum corrections, this root variance going as $\sim 1/\sqrt{\mathcal{N}}$).

Also, we want to discuss the Non Relativistic (NR) limit, so it will be useful to decompose the field as

$$\phi = \frac{1}{\sqrt{2m}}\left(\psi\mathrm{e}^{-\mathrm{i}mt} + \psi^*\mathrm{e}^{\mathrm{i}mt}\right) ; \tag{3.7}$$

ψ is a complex scalar field, with the dimensions of a mass square (since the real scalar field ϕ must have the dimensions of a mass). Put the previous in (3.5), neglect terms exploiting the NR limit condition $|\dot{\psi}| \ll m|\psi|$, and obtain the Schrödinger-like equation

$$\mathrm{i}\partial_0\psi + \frac{3}{2}\mathrm{i}H\psi = -\frac{\nabla^2\psi}{2ma^2} + m\Phi\psi . \tag{3.8}$$

In the following, since we are discussing galactic dynamics (length scale $\sim 10\,\mathrm{kpc}$), we can safely neglect the Hubble term and universe expansion effects, so we will set $a = 1$ and $H = 0$.

To obtain the equation for the gravitational potential, we can use the 00 Einstein equation in the weak field limit:

$$\nabla^2\Phi = 4\pi G T_{00} ; \tag{3.9}$$

the energy momentum tensor reads

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\left(\frac{1}{2}g^{\gamma\delta}\partial_\gamma\phi\partial_\delta\phi + \frac{1}{2}m^2\phi^2\right) , \tag{3.10}$$

so $T_{00} = |\psi|^2$ in the NR limit.

In the end, we end up with the Schrödinger-Poisson equations

$$\mathrm{i}\partial_0\psi = -\frac{\nabla^2\psi}{2m} + m\Phi\psi , \tag{3.11}$$

$$\nabla^2\Phi = 4\pi G(|\psi|^2 - \langle|\psi|^2\rangle) . \tag{3.12}$$

4 Features of ULDM

Cores The ground state solution of the Schrödinger-Poisson equations is a cored solution. To see this, decompose

$$\psi(\vec{x},t) = \frac{mM_{\rm pl}}{\sqrt{4\pi}}\mathrm{e}^{-\mathrm{i}\gamma mt}\chi(\vec{x}) , \tag{4.1}$$

where γ is an eigenvalue of the problem. Assume spherical symmetry and define the adimensional radius $x := rm$, then the Schrödinger-Poisson equations become

$$\partial_x^2\chi + \frac{2}{x}\partial_x\chi = 2(\Phi - \gamma)\chi , \tag{4.2}$$

$$\partial_x^2\Phi + \frac{2}{x}\partial_x\Phi = \chi^2 . \tag{4.3}$$

The ground energy state solution (called also soliton) correspond to the one with no nodes; it is possible to easily solve these equations numerically, by assuming $\chi(0) = \lambda_{\rm s}^2$ as initial condition, together, for consistence with the form of the equations, with $\partial_x\chi = \partial_x\Phi = 0$; the initial condition for $\Phi(0)$ can be found by selecting the value that allows for a no-node solution with $\chi(x \to \infty) \to 0$, whereas γ is fixed by imposing $\Phi(x \to \infty) \to 0$. Numerical calculations show that, for $\lambda_{\rm s}^2 = 1$, $\gamma_1 \simeq -0.69$, where the subscript 1 indicates the solution corresponding to the initial condition $\lambda_{\rm s}^2 = 1$. It is easy to see that, for a generic $\lambda_{\rm s}$, the solutions scale as

$$\chi_{\lambda_{\rm s}}(x) = \lambda_{\rm s}^2\chi_1(\lambda_{\rm s}x) , \tag{4.4}$$

$$\Phi_{\lambda_{\rm s}}(x) = \lambda_{\rm s}^2\Phi_1(\lambda_{\rm s}x) , \tag{4.5}$$

$$\gamma_{\lambda_{\rm s}} = \lambda_{\rm s}^2\gamma_1 ; \tag{4.6}$$

in particular, defining the mass of the χ_1 soliton as

$$M_1 = \int \mathrm{d}^3r\,|\psi^2| = \frac{M_{\rm pl}^2}{m}\int_0^\infty \mathrm{d}x\,x^2\chi_1^2(x) \simeq 2.79\times10^{10}\left(\frac{m}{10^{-20}\mathrm{eV}}\right)^{-1}M_\odot \tag{4.7}$$

and the core radius as as the one where the mass density drops by half, yielding

$$r_{\rm c1} \simeq 8.2\times10^{-5}\left(\frac{m}{10^{-20}\,\mathrm{eV}}\right)^{-1}\mathrm{pc} , \tag{4.8}$$

we have $M_\lambda = \lambda_{\rm s}M_1$, $r_{{\rm c}\lambda} = \lambda_{\rm s}^{-1}r_{\rm c1}$, so their product is independent of $\lambda_{\rm s}$, and yields

$$M_\lambda r_{{\rm c}\lambda} \simeq 2.27\times10^6\left(\frac{m}{10^{-20}\,\mathrm{eV}}\right)^{-2}\mathrm{kpc}\,M_\odot , \tag{4.9}$$

Exploiting equation (4.7), we can write

$$\lambda_{\rm s} \simeq 3.6\times10^{-2}\left(\frac{m}{10^{-20}\,\mathrm{eV}}\right)\left(\frac{M_\lambda}{10^9M_\odot}\right) \tag{4.10}$$

It can be useful to write down an analytic approximation for the soliton density. It is

$$\rho_\lambda(r) = \frac{190\left(\frac{10^{-20}\,\mathrm{eV}}{m}\right)^2\left(\frac{10\,\mathrm{pc}}{r_{{\rm c}\lambda}}\right)^4}{\left(1 + 0.091\left(\frac{r}{r_{{\rm c}\lambda}}\right)^2\right)^8}M_\odot\mathrm{pc}^{-3} . \tag{4.11}$$

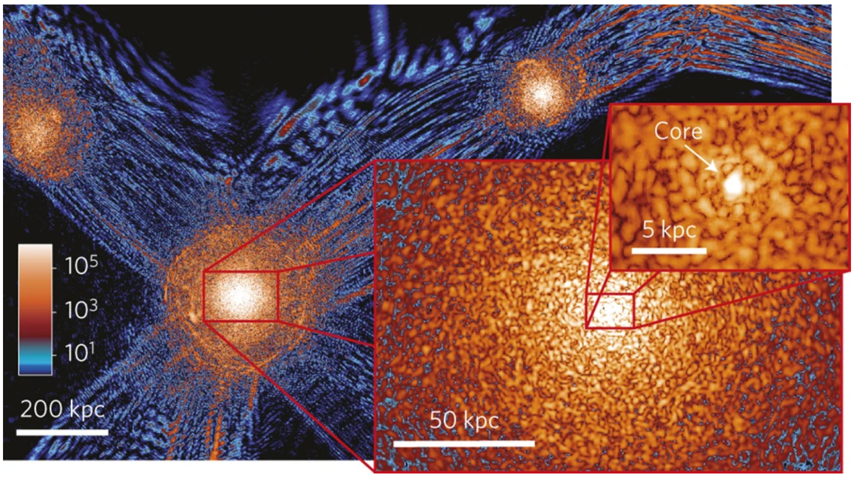

Figure 1: An example of an ULDM simulation from Schive, Chiueh, and Broadhurst 2014. Both the formation of cores and the interference behaviour coming from the wave nature of ULDM are apparent.

The energy can be computed as, with $x' := m\lambda_\mathrm{s} r$

$$\begin{aligned} E_{\lambda_\mathrm{s}} =& \frac{m^2 M_\mathrm{PL}^2}{4\pi}\lambda_\mathrm{s}^6 \int \frac{\mathrm{d}^3 x'}{m^3\lambda_\mathrm{s}^3}\left(\frac{1}{2}\left|\boldsymbol{\nabla}'\chi_1(x')\right|^2 + \frac{\Phi_1(x')|\chi_1|^2(x')}{2}\right) \\ &\implies E_{\lambda_\mathrm{s}} = \lambda_\mathrm{s}^3 E_1 = \frac{\lambda_\mathrm{s}^3}{3}\gamma_1 M_1 \ , \end{aligned} \tag{4.12}$$

(to understand this expression, recall that in (3.7) we factored out a mass term, giving ψ the dimensions of a mass squared, and that the gravitational potential is a n-body interaction one, hence the factor 2 at the denominator in the potential term); the last equality is obtained using the virial theorem, $E_\mathrm{p} = -2E_\mathrm{k}$, valid for the stationary Schrödinger-Poisson.

Fluctuating density field The validity of the wave description implies that the density field itself experiences wave phenomena like interference. In general, a wavefunctions can be expressed as a superposition of eigenmodes (we can take a_j to be real)

$$\psi(\vec{x},t) = \sum_j a_j\psi_j(\vec{x})\mathrm{e}^{\mathrm{i}\varphi_j}\mathrm{e}^{-\mathrm{i}\omega_j t} \implies \rho(\vec{x}) = |\psi|^2 = \sum_j |a_j|^2|\psi_j(\vec{x})|^2 + \sum_{j\neq k} a_j a_k \psi_j\psi_k^* \mathrm{e}^{\mathrm{i}(\varphi_j-\varphi_k)}\mathrm{e}^{-\mathrm{i}t(\omega_j-\omega_k)} \ , \tag{4.13}$$

where the first term is the average density, and the last term is the interference term. The formation of cores and this typical interference pattern are seen in simulations, see e.g. Fig. 1.

"Quantum pressure" and small scales suppression We can recast the Schrödinger-Poisson as continuity and Euler equation from fluid dynamics, using the so-called Madelung formulation

$$\psi = \sqrt{\frac{\rho}{m}}\mathrm{e}^{\mathrm{i}\theta} \ , \ \vec{v} = \frac{1}{m}\boldsymbol{\nabla}\theta = \frac{1}{2m|\psi|^2\mathrm{i}}(\psi_i^*\boldsymbol{\nabla}\psi_i - \psi_i\boldsymbol{\nabla}\psi_i^*) \ , \tag{4.14}$$

to obtain

$$\partial_t\rho + \boldsymbol{\nabla}\cdot(\rho\vec{v}) = 0 \ , \tag{4.15}$$

$$\partial_t\vec{v} + (\vec{v}\cdot\boldsymbol{\nabla})\vec{v} = -\boldsymbol{\nabla}\Phi + \frac{1}{2m^2}\boldsymbol{\nabla}\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right) \ . \tag{4.16}$$

Such a formulation does not describe vortices, which are configurations where the density vanishes in one point (where the Madelung formulation Eq. (4.14) loses its meaning).

Notice that Eqs.(4.15)(4.16) are completely classical, and that we can loosely identify, in analogy with how the Euler equation looks like,

$$\frac{1}{2m^2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}} \sim P_\mathrm{quantum} \ , \tag{4.17}$$

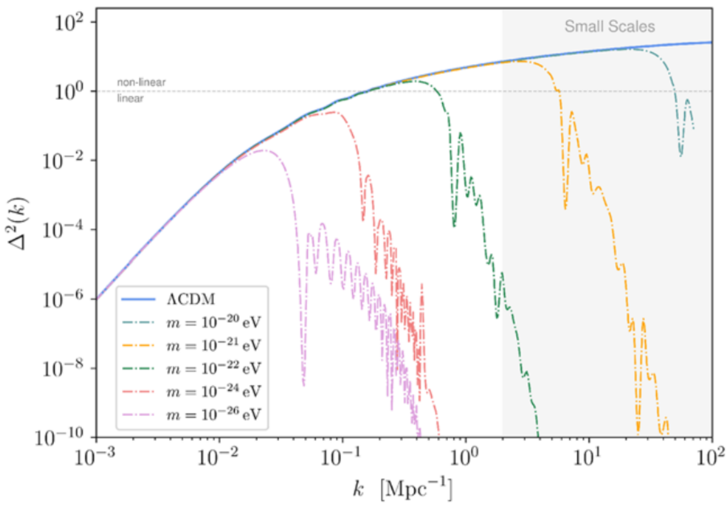

Figure 2: Example of small scale suppression power expected from ULDM, from Ferreira 2021

which is usually called quantum pressure. This is unfortunately a misnomer on multiple levels. One, as we remarked, the equations are classical, so there is no real quantum behaviour. Second, it is not really a pressure, since it does not come from the isotropic part of the T_{ij} tensor. Nevertheless, this name gives intuition on its effects: it acts like an outward pressure, which slows down gravitational collapse, and whose origin can be traced back on the uncertainty principle (I cannot squish too much matter in a small volume: the more I do, the more the position of particles becomes precise, and hence the more the velocity dispersion has to increase).

To understand at least qualitatively this effect of resistance to gravitational collapse (which will end up in small scales power spectrum suppression), we can consider to expand Eqs.(4.15)(4.16) in $\rho = \bar{\rho}(1+\delta)$, with $|\delta| \ll 1$, and $|\vec{v}| \ll 1$. At first order in δ and $\vec{v}$, we have

$$\dot{\vec{v}} = -\boldsymbol{\nabla}\Phi + \frac{1}{2m^2}\boldsymbol{\nabla}\frac{\nabla^2\delta}{2} \ , \ \dot{\delta} + \boldsymbol{\nabla} v = 0 \ ; \tag{4.18}$$

taking the derivative of the latter and using the former, together with the Poisson equation[1]

$$\nabla^2\Phi = 4\pi G\bar{\rho}\delta \ , \tag{4.19}$$

we have

$$\ddot{\delta} - 4\pi G\bar{\rho}\delta + \frac{1}{4m^2}\boldsymbol{\nabla}^4\delta = 0 \ . \tag{4.20}$$

The previous is solved by

$$\delta = A_1 \mathrm{e}^{\mathrm{i}(\omega t - \vec{k}\cdot\vec{x})} + A_2 \mathrm{e}^{-\mathrm{i}(\omega t - \vec{k}\cdot\vec{x})} \ , \tag{4.21}$$

with

$$\omega^2 = \frac{1}{4m^2}k^4 - 4\pi G\bar{\rho} \ ; \tag{4.22}$$

this defines the so called Jeans scale

$$k_{\mathrm{J}} = (16\pi G\bar{\rho})^{1/4} m^{1/2} \ . \tag{4.23}$$

On perturbations with scales $k < k_{\mathrm{J}}$, gravity dominates, ω is imaginary and we have a exponentially increasing mode (gravitational collapse), whereas for scales $k > k_{\mathrm{J}}$, quantum pressure dominates, the solutions in Eq. (4.21) are just waves, and collapse is halted.

Fig. 2 shows this small scales suppression effect for different ULDM masses. This effect is an example of an ULDM feature which allows us to set constraints on m. In particular, $m > 1 \times 10^{-22}$ eV if ULDM comprises the whole DM content of the universe.

[1]The Newtonian potential Φ is sourced only by the fluctuations. The mean density $\bar{\rho}$ sources the expansion of the universe (in Friedmann equations).

References

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