

Weizmann Institute of Science

Department of Particle Physics & Astrophysics

PhD in Physics

Final Dissertation

**Gravitational Probes of Ultralight Dark Matter and
Degeneracies in Time-Delay Cosmography**

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Declaration of Authorship

I, Luca TEODORI, declare that this thesis titled, “Gravitational Probes of Ultralight Dark Matter and Degeneracies in Time-Delay Cosmography” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this Institute.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

This thesis contains results from 4 published works and 2 additional submitted works:

- Kfir Blum and Luca Teodori, “Hunting for ultralight dark matter with cosmographic H_0 signal”, arXiv:2409.04134, published in: Phys.Rev.D. 111 (2025) 4, 043509.

Main author, I performed the ultralight dark matter simulations and produced most of the main results.

- Luca Teodori and Kfir Blum, “Host group degeneracy in gravitational lensing time delay determination of H_0 ”, arXiv:2305.19151, published in: JCAP 08 (2023) 065.

Main author, I did all the MCMC inferences and found the discrete degeneracy discussed in the paper.

- Luca Teodori, Kfir Blum, Emanuele Castorina, Marko Simonović and Yotam Soreq, “Comments on the mass sheet degeneracy in cosmography analyses”, arXiv:2201.05111, published in: JCAP 07 (2022) 07, 027.

Main author, I worked out the weak lensing effects and the observable in multiple source systems; I used CAMB to compute the differential external convergence expectations.

- Kfir Blum and Luca Teodori, “Gravitational lensing H_0 tension from ultralight axion galactic cores”, arXiv:2105.10873, published in: Phys.Rev.D 104 (2021) 12, 123011.

Main author, I did most of the calculations and estimates; I contributed to the Python package `lenstronomy` and did the lens mock analysis.

- Luca Teodori, Andrea Caputo and Kfir Blum, “Ultra-Light Dark Matter Simulations and Stellar Dynamics: Tension in Dwarf Galaxies for $m < 5 \times 10^{-21}$ eV”, arXiv:2501.07631, submitted to PRL.

Main author, I performed the numerical simulations and produced most of the main results.

- Kfir Blum, Marco Gorghetto, Edward Hardy, Luca Teodori, “Bracketing the soliton–halo relation of ultralight dark matter”, arXiv:2504.16202, submitted to JCAP.

Main author, I performed the numerical simulations, produced all the plots and most of the main results.

WEIZMANN INSTITUTE OF SCIENCE

Abstract

Faculty of Physics

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Gravitational Probes of Ultralight Dark Matter and Degeneracies in Time-Delay Cosmography

by Luca TEODORI

The nature of dark matter remains one of the big unsolved questions in physics. Furthermore, the standard cosmological model, Λ CDM, is facing several challenges, most notably the H_0 tension. Strong gravitational lensing and stellar kinematics (strong lensing as a probe of cosmology and both as probes of dark matter) offer promising avenues to advance our understanding of these fundamental topics through gravitational phenomena. In particular, UltraLight Dark Matter (ULDM), a well-motivated and highly studied dark matter candidate, can be constrained using gravitational probes alone.

In this thesis, we primarily address two key topics: how strong gravitational lensing and stellar kinematics can be used as gravitational probes of ULDM, and the degeneracies that arise when using strong gravitational lensing as a cosmological probe, particularly for measuring H_0 via time-delay cosmography.

We show, both theoretically and through numerical simulations, that ULDM in the mass range $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$, with total dark matter budget fractions $\alpha_\chi \lesssim 0.15$, can form cores that are detectable in strong gravitational lensing systems, particularly in time-delay cosmography. This region of the parameter space remains unconstrained, and the wealth of data from ongoing and future missions will provide opportunities to probe this scenario.

We explore the use of stellar kinematics in galaxies, particularly dwarf galaxies, as an additional gravitational probe of ULDM through the mechanism of dynamical heating of stars. Using numerical simulations, we show that the stellar kinematics and distribution of dwarf galaxies are in tension with observational data for $m \lesssim 5 \times 10^{-21} \text{ eV}$.

We examine the soliton-halo relation, which provides predictions for the structure of stellar rotation curves in galaxies. Our results show that most solitons are sufficiently massive to produce an inner rotation curve velocity peak that exceeds the outer rotation curve peak, the latter from the halo. Additionally, we provide insights that may help reconcile the differing results found in the literature regarding the soliton-halo relation.

In the context of degeneracies in time-delay cosmography, we highlight the necessity of accounting for all external convergences along the sightlines, particularly the observer-lens sightline, to achieve an unbiased measurement of H_0 . We clarify how multiple source systems can aid in breaking the mass sheet degeneracy, a well-known challenge in time-delay cosmography, and discuss the role of differential external convergences in potentially masking this breaking effect. Finally, we identify a discrete approximate degeneracy that arises in strong lens systems requiring explicit modelling of the host group halo. If observed systems are influenced by this degeneracy, the measurement of H_0 may be biased upward.

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Symbols

$\vec{}$	3D spatial vector or 2D angle vector	3
δ_{D}	Dirac delta	3
$a(t)$	FRW cosmological scale factor	3
δ	Cosmological matter density fluctuation	4
κ	Curvature parameter	4
η	Conformal time	4
w	comoving metric distance	4
$f_\kappa(w)$	Comoving distance	4
Λ	Cosmological constant	5
u^μ	4-velocity	5
w_s	Equation of state parameter	5
$\Omega_{j,0}$	Cosmological density parameter of component j today	5
z	Redshift	6
D_A	Angular diameter distance	6
D_L	Luminosity distance	7
ζ_δ	Density correlator	8

- P_δ Power spectrum 8
- Φ Gravitational potential 9
- $D_\pm$ Growing/decaying perturbation mode 10
- $\vec{\psi}$ Lagrangian displacement field 11
- n_* Number density of galaxies/stars/tracers 15
- $\beta(r)$ Anisotropy parameter 15
- σ_{los} Line of sight velocity dispersion 16
- ζ^μ Geodesic displacement vector 20
- s_A^μ Screen vector 20
- Θ Heaviside function 21
- $\Psi(\vec{\theta})$ Lensing potential 22
- D_l Lens angular diameter distance 22
- D_{ls} Lens-source angular diameter distance 22
- D_s Source angular diameter distance 22
- $\vec{\alpha}$ Displacement angle 22
- $\kappa(\vec{\theta})$ Lensing convergence 22
- $\Sigma(\vec{\theta})$ Column density 22
- Σ_{cr} Lensing critical column density 22
- r_{Sc} Schwarzschild radius 23
- Γ Euler gamma function 24
- θ_E Einstein angle 24
- D_{dt} Time delay distance 25
- Φ_t Gravitational potential from large scale structure in the tidal approximation 26
- Y_{lm} Spherical harmonics 29
- j_l Spherical Bessel functions 29
- ψ Ultralight dark matter non-relativistic field 32
- χ_λ Soliton wavefunction 34
- R_s Navarro-Frenk-White profile scale radius 48
- $\Gamma_{\mu\nu}^\rho$ Christoffel symbol 161
- $R^\mu_{\alpha\beta\gamma}$ Riemann tensor 161
- $R_{\mu\nu}$ Ricci tensor 161
- $g_{\mu\nu}$ Metric coefficients 161
- R Ricci scalar 161
- $T_{\mu\nu}$ Energy-momentum tensor 161

∇_μ Covariant derivative 161

Acronyms

Λ CDM “ Λ ” Cold Dark Matter 1, 2, 3, 4, 11, 14, 17, 125

BAO Baryon Acoustic Oscillations 13, 17

CDM Cold Dark Matter 1, 2, 8, 10, 14, 31, 37, 75, 85, 86, 87, 88, 90, 91, 93, 95, 97, 102, 104

CMB Cosmic Microwave Background 4, 14, 17, 18, 31, 37, 47, 82, 83, 85, 95, 125

DFT Discrete Fourier Transform 133, 134, 135, 137

DM Dark Matter 1, 2, 14, 15, 19, 31, 32, 33, 34, 37, 39, 40, 75, 77, 78, 80, 85, 87, 95, 101, 103, 106, 109

FFT Fast Fourier Transform 135

FLRW Friedmann-Lemaître-Robertson-Walker 4, 6, 19, 20, 81, 149, 153, 156, 162, 165, 167

FWHM Full Width Half Maximum 41

GR General Relativity 3, 161

LHS Left Hand Side 5, 67, 166, 182

LOS Line Of Sight 2, 26, 30, 39, 43, 44, 51, 60, 61, 69, 70, 71, 72

LOSVD Line Of Sight Velocity Dispersion xi, 15, 48, 52, 57, 58, 61, 66, 97, 98, 99, 101, 102, 103, 104, 105, 106, 107

LSS Large Scale Structure 1, 2, 4, 8, 14, 17, 18, 19, 26, 27, 28, 37, 39, 62, 63, 68, 127

MCMC Markov Chain MonteCarlo 44, 48, 50, 51, 59, 61, 83, 84, 86, 126, 181

MSD Mass Sheet Degeneracy 2, 3, 18, 19, 25, 26, 39, 40, 41, 43, 44, 61, 62, 63, 64, 66, 67, 72, 75, 77, 78, 83, 85, 88, 90, 92, 95, 125, 126, 135, 179, 181, 182

NFW Navarro-Frenk-White 14, 15, 43, 44, 47, 48, 50, 52, 57, 58, 59, 60, 61, 76, 85, 87, 88, 90, 91, 93, 98, 101, 103, 104, 106, 108, 110, 111, 114, 117, 183, 185

NR Non Relativistic 32, 33

PL Power Law 24, 40, 51, 52, 77, 78, 79, 80, 81, 82, 83, 84, 179

PSF Point Spread Function 41

RHS Right Hand Side 26, 58, 166, 182

SIS Singular Isothermal Sphere 52, 57

SN Ia SuperNovae Ia 17, 72

SPE Schrödinger-Poisson Equations 31, 33, 35, 36, 98, 109, 119, 129, 133, 135, 136

TRGB Tip of the Red Giant Branch 17

ULDM UltraLight Dark Matter 2, 3, 31, 32, 33, 34, 35, 37, 40, 66, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 97, 98, 100, 101, 102, 103, 104, 107, 109, 110, 111, 117, 121, 123, 125, 126, 127, 129, 131, 135, 144, 145, 162, 166, 179

Chapter 1

Introduction

When we look at galaxies and at the universe on the largest scales, we see the gravitational influence of more matter than the one we can actually see or account for with standard model of particle physics alone. In particular, the standard Λ CDM model, which is our best and simplest model which can explain the universe on the large scales, predicts this extra matter to be about 5 times more abundant than the known matter (“baryons”). The nature of this missing matter, or Dark Matter (DM), is yet to be uncovered.

So far, we have evidence of DM only via its gravitational influence. One of the first evidences for the existence of DM came in the early 1930’s, from Oort observations of star motions in the Milky way [Oor] and Zwicky observations of galaxy velocity dispersions in the Coma cluster [Zwi33; Zwi79], both hinting at much more mass than what one would expect from visible matter. The observation that made the idea of DM dominant in the physics literature came from Ref. [Rub83] work on velocity curves of galaxies. After that, we gathered many more evidence of the presence of DM, from gravitational lensing [WCW79; LP89; BPL90], from cosmology and Large Scale Structure (LSS), galaxy clusters [Eis+05; Agh+20], from which we understood that DM is really weakly coupled to standard model particles, if at all, and that it has to be “cold”. However, as of now, we do not have a clear idea on what DM really is.

The challenge lies in the fact that there exists an enormous variety of DM candidates capable of explaining most, if not all, of the observed phenomena; specifically, DM candidates that behave like Cold Dark Matter (CDM) on large scales [BT18]. However, CDM is unlikely to provide the complete picture, as the CDM paradigm faces several shortcomings, particularly on small (i.e., astrophysical) scales [BBK17]. Furthermore, no existing theory currently connects CDM to other known particles within a unified framework.

The nature of dark matter is just one of the many questions arising in the context of the Λ CDM model. Within Λ CDM, one of the most discussed cosmological tensions in the literature is the Hubble, or H_0 , tension [DV+21; VTR19; VSGM23; FM23; DV+25]. The H_0 tension is the discrepancy on the measured value of H_0 , which quantifies the expansion rate of the universe, between different probes. In particular, probes based on early universe physics and/or LSS predict $H_0 \sim 67 \text{ km Mpc}^{-1} \text{ s}^{-1}$ [Agh+20; Bal+23; Aio+20; Ala+21; D’A+20; ISZ20], on the low side. Probes based on late universe physics, most notably the Cepheid distance ladder-based measurements, give $H_0 \sim 73 \text{ km Mpc}^{-1} \text{ s}^{-1}$ [Rie+16; Rie+21; Rie+22], on the high side. There are heavy investigations on possible measurement systematics or different late-universe probes which may explain away such a tension [Cal+08; FM23; Fre+24; DV+25]. There are also a plethora of theoretical models trying to explain such discrepancy, assuming the tension is real [DV+21].

A promising gravitational probe for the nature of dark matter and also for measuring H_0 , major focus of this thesis, comes from gravitational lensing.

As far as H_0 is concerned, gravitational lensing offers a way to measure H_0 via time delays between multiple image lens systems. Such a measurement is independent of the aforementioned early universe

probes and distance-ladder based methods; it is a late universe measurement as the distance-ladder one, and it can provide a cross-check for whether the H_0 tension is real or not. As any measurement of this type, the H_0 measurement from gravitational lensing suffers from systematic biases. The most famous systematic bias comes from the Mass Sheet Degeneracy (MSD), a well-known mathematical degeneracy in the gravitational lens equations [FGS85] involving the lens density profile and time-delays. Physical realizations of such a degeneracy include internal DM core components, the DM halo of the group/cluster the lens galaxy resides on, and the LSS on the Line Of Sight (LOS). The study of this degeneracy is a main point of the works presented in this thesis. In Ch. 4, we explore all these possible physical realizations of MSD, and discuss some key aspects of their influence in time-delay cosmography. In particular, we show in Sec. 4.3.2 how the perfect MSD limit adopted in some analyses [Bir+20] is not conservative as far as stellar kinematics is concerned. In Sec. 4.2, we discover and discuss a discrete degeneracy involving the flexion field of the DM halo of the group which hosts the main lens; in Sec. 4.3, we point out the necessity of accounting for the observer-lens sightline external convergence to achieve an unbiased measurement of H_0 , and discuss how multiple source systems can aid in breaking the MSD, and how differential external convergences can mask this breaking effect.

On the dark matter side, strong gravitational lensing is sensitive to the dark matter density profile in galaxies [Veg+23]. Among the CDM candidates which have interesting observable features which can be probed from gravitational lensing, UltraLight Dark Matter (ULDM) is the one we will focus on in this thesis. ULDM is an highly discussed DM candidate in the literature [HBG00; Hui+17; Hui21; Fer21], as it has phenomenological effects which can be probed via gravity alone. Its most defining feature is the fact that the de-Broglie wavelength associated to its very small mass is order parsec or larger, i.e. order of astrophysical scales. In particular, ULDM is better described as a wave rather than a particle. One key predictions of the ULDM paradigm is the formation of cored profiles. In the context of gravitational lensing, such cores act as approximate MSD. The presence of unaccounted ULDM cores in galaxies can affect the lensing H_0 measurement or, equivalently, they can affect time delays if an H_0 prior is used when interpreting data. With the latter point of view in mind, we will show how ULDM cores can be searched for, using strong gravitational lensing systems and time delays, in Ch. 5. In particular, we find that ULDM can be constrained via time-delay cosmography in the mass range $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$, with total DM budget fractions between 5% and 15%. This is a region which is as of now unconstrained. We dub this way of searching for ULDM AxionH0graphy.

The particular features of ULDM affect also stellar kinematics and dynamics in galactic systems, as we will discuss in Ch. 6. We show how dynamical heating of stars, coming from ULDM density fluctuating field (which in turn is a manifestation of the wave behaviour of ULDM) can be used to put constraints on the allowed particle mass of ULDM. We show this via numerical simulations of dwarf galaxy systems in a ULDM halo. In particular, in Sec. 6.1, we show how ULDM with $m \lesssim 5 \times 10^{-21}$ eV is in tension with dwarf galaxy star kinematics and star distribution data.

Stellar kinematics is also affected by ULDM cores, which in the literature are known as solitons. The amount of mass of a soliton, forming in the potential well of a given DM halo, will determine the stellar circular velocity curve structure of a typical ULDM halo. The relation between the mass of the soliton and the features of its host halo is called soliton-halo relation. We investigate the soliton-halo relation via numerical simulations and show how generically the circular velocity curve of a typical ULDM halo has two peaks, one from the soliton and one from the halo, the former usually higher than the latter. In particular, we put forth a new point of view on the soliton-halo relation, which may reconcile various results in the literature, as we explain in Sec. 6.3.

To summarize, out of the problems arising within Λ CDM, in this thesis we focus on the H_0 tension, in the context of its measurement via gravitational lensing time delays; in particular, we focus on degeneracies in time-delay cosmography, most notably the MSD. Out of the possible DM candidates, we instead focus on ULDM, and investigate it in the context of time-delay cosmography and stellar kinematics/dynamics in galaxies.

This thesis is organized as follows. We review the basics of the Λ CDM model in Sec. 1.1 and of stellar kinematics probes of dark matter in Sec. 1.2. We review the status of the H_0 tension in Sec. 1.3. In Ch. 2 and Ch. 3, we introduce the fundamental concepts of gravitational lensing and of ULDM respectively. We introduce an approximation for the calculation of external convergence expectation values in Sec. 2.4 relying on results we show in Sec. 1.1.6, a novel result of this thesis from Ref. [Teo+22].

The main body of the thesis, showing novel results from [BT21b; Teo+22; TB23; BT25; TCB25; Blu+25], is in Ch. 4, Ch. 5 and Ch. 6. We discuss the impact of MSD in time-delay cosmography in Ch. 4 and how one can exploit time delays as a probe of ULDM in Ch. 5. We describe our efforts in using stellar kinematics and dynamics to probe ULDM via numerical simulations in Ch. 6, where we investigate ULDM dynamical heating in Sec. 6.1 and the soliton-halo relation with its implications in Sec. 6.2, Sec. 6.3. We finally summarize all the main results of this thesis in Ch. 7.

As a project which, at the moment of writing, is work in progress, we show a novel calculation of time-delays in the Kottler metric in App. B.

Most of the code and notebooks used to produce the main results shown can be found on the github page <https://github.com/lucateo/>.

1.0.1 Conventions

We will use the following convention for the Fourier transform

$$f(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} f(\vec{k}) , \quad (1.1)$$

where we indicate the Fourier transform of f simply by specifying its dependence on $\vec{k}$. d is the dimension of $\vec{x}$. In this thesis, $\vec{\cdot}$ denotes 3D spatial vector quantities or 2D vector angles, depending on context.

Conversely, the inverse Fourier transform will be

$$f(\vec{k}) = \int d^d x e^{-i\vec{k} \cdot \vec{x}} f(\vec{x}) , \quad (1.2)$$

and thus the integral representation of the Dirac delta δ_D will be

$$\delta_D(\vec{x}) = \int \frac{d^d k}{(2\pi)^d} e^{i\vec{k} \cdot \vec{x}} . \quad (1.3)$$

We will always use, unless it is clearly specified otherwise, the Einstein summation convention, i.e. summation over repeated indices, that appear repeated only in one side of an equation, is understood.

We will use the General Relativity (GR) sign conventions as in Ref. [TMW00], detailed in App. C.1. In particular, we use the mostly positive metric signature.

We will use natural units, with $c = \hbar = 1$. We will convert to standard units (the one pertaining the international system of units or usual astronomical units like Mpc) when we will do quantitative analyses.

We will normalize the Robertson-Walker scale factor $a(t)$ to be unity at the time t_0 corresponding to today, $a_0 := a(t_0) = 1$.

1.1 Essentials of the Λ CDM model

The Λ CDM model is as of now our best model which can explain most of our observations of the universe on the large scales, from the Cosmic Microwave Background (CMB) to LSS observables.

One of the key assumptions of the Λ CDM model is that on sufficiently large scales, the universe is isotropic (i.e. the universe looks the same in every line of sight) and homogeneous (the properties of the universe do not depend on the particular point in space we are). In other words, the universe is rotation and translation invariant (but critically, not invariant under time translations).

If we look at the very early universe through the CMB, or we look at very large scales, this assumption seems justified. In particular, if we split the matter-energy density $\rho(\vec{x}, t)$ as

$$\rho(\vec{x}, t) = \bar{\rho}(t)(1 + \delta(\vec{x}, t)), \quad (1.4)$$

where $\bar{\rho}$ is the average density at comoving time t , also called background density, and δ is the fluctuation over such a background, then around the recombination times, when CMB photons were finally able to free-stream all the way to us, we have $\delta \ll 1$ (we can infer this from the very small temperature fluctuations in the CMB). What this implies is that we can describe the early universe as a perturbed homogeneous and isotropic one, where homogeneous and isotropic is the zeroth order approximation. In the next section, we describe the equations of an homogeneous and isotropic universe.

1.1.1 The Friedmann equations

Upon an isotropic and homogeneous universe, the most general line element is the Friedmann-Lemaître-Robertson-Walker (FLRW) one [Fri22; Fri24; Lem13; Rob35; Wal37; Wei08], which can be written as

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2(t) \left(\frac{dr^2}{1 - \kappa r^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \right). \quad (1.5)$$

κ is the curvature parameter describing the overall geometry of the universe and $a(t)$ is the cosmological scale factor, encoding the expansion of the universe; we have a flat, spherical/closed or hyperbolic/open universe for $\kappa = 0$, $\kappa > 0$ or $\kappa < 0$ respectively. Our universe seems best described as a flat one, $\kappa \simeq 0$.

We can write Eq. (1.5) by using the conformal time $d\eta = dt/a(t)$ and the comoving coordinate w ; defining

$$r = f_\kappa(w) = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}w), & \kappa > 0; \\ w, & \kappa = 0; \\ \frac{1}{\sqrt{-\kappa}} \sinh(\sqrt{-\kappa}w), & \kappa < 0; \end{cases} \quad (1.6)$$

we end up with the simpler (with $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$)

$$ds^2 = a^2(\eta)(-d\eta^2 + dw^2 + f_\kappa^2(w)d\Omega^2), \quad (1.7)$$

which is a flat metric multiplied by the conformal factor $a(\eta)$. We have the freedom to set the scale factor at the present time as $a(t_0) = 1$, thanks to the invariance of Eq. (1.7) under the rescaling

$$a \rightarrow \lambda a, \quad w \rightarrow w/\lambda, \quad \kappa \rightarrow \lambda^2 \kappa. \quad (1.8)$$

Notice that the sign of κ remains unchanged under the previous rescaling.

We determined the metric, which enters the LHS of Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} - \Lambda g_{\mu\nu} , \quad (1.9)$$

where Λ is the cosmological constant; we thus need now to determine the energy-momentum tensor $T_{\mu\nu}$. The proper $T_{\mu\nu}$ obeying homogeneity and isotropy is the perfect fluid energy-momentum tensor, which reads

$$T_{\mu\nu} = \bar{P}(t)g_{\mu\nu} + (\bar{P}(t) + \bar{\rho}(t))u_\mu u_\nu , \quad (1.10)$$

where u^μ is the four velocity of the observer, and $\bar{\rho}$, $\bar{P}$ are the background density and pressure respectively, which depend on the comoving time t alone. A comoving observer has $u^\mu = (1, 0, 0, 0)$. Using the metric Eq. (1.7), together with the perfect fluid energy tensor Eq. (1.10), we obtain the Friedmann equations (see Ch. C for derivation; $\dot{} := d/dt$):

$$\left(\frac{\dot{a}}{a}\right)^2 =: H^2 = \frac{8\pi G}{3}\bar{\rho} - \frac{\kappa}{a^2} + \frac{\Lambda}{3} , \quad (1.11)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{P}) + \frac{\Lambda}{3} ; \quad (1.12)$$

the third Friedmann equation, also called continuity equation, can be derived from the previous two or also from the zeroth component of the continuity equation $\nabla_\nu T^{\mu\nu} = 0$, and reads

$$\dot{\bar{\rho}} + 3H(\bar{\rho} + \bar{P}) = 0 . \quad (1.13)$$

Notice that κ appears in equations always in the combination κ/a^2 , which is invariant under the rescaling Eq. (1.8).

Different components of the universe are characterized by a different equation of state $\bar{P} = w_s \bar{\rho}$, in particular $w_s = 0$ for matter and $w_s = 1/3$ for radiation; with specified w_s , we can express the first Friedmann equation Eq. (1.11) in terms of the various density parameters today

$$\Omega_{j,0} := \frac{8\pi G}{3H_0^2} \bar{\rho}_{j,0} , \quad (1.14)$$

where we used 0 as a subscript for quantities evaluated at the present time, and the index j labels the various components (r = radiation, m=matter, md= (cold) dark matter, mb = baryons, Λ = dark energy, with the definition $\rho_{\Lambda,0} := \Lambda/8\pi G$). Using the third Friedmann equation together with the equation of state, we have

$$\bar{\rho}_j(t) = \bar{\rho}_j(t_0)a(t)^{-3(1+w_s)} . \quad (1.15)$$

With this, the first Friedmann equation Eq. (1.11) becomes

$$H^2 = H_0^2 \left(\Omega_{r,0}a^{-4} + \Omega_{m,0}a^{-3} + \Omega_{\kappa,0}a^{-2} + \Omega_{\Lambda,0} \right) , \quad (1.16)$$

where $\Omega_{\kappa,0}/a^2 = -\kappa/(H_0^2 a^2)$ parametrizes how far are we from a flat universe. Our universe so far seems compatible with $\Omega_{r,0} \leq 10^{-5}$, $\Omega_{\kappa,0} \sim 0$, $\Omega_{m,0} \sim 0.3$, $\Omega_{\Lambda,0} \sim 0.7$ [Agh+20]. In the standard hot big bang model, the universe in its early stage was radiation dominated, $\Omega_r \sim 1$; the different power-law decay behaviour of different components with the expansion makes the universe go towards different phases, from the early radiation domination (which decays as a^4) towards matter domination (which decays as a^3) where Ω_m dominates, to the era we are in today, dark energy domination.

1.1.2 Distances in cosmology

The concept of distance in cosmology, due to expansion of the universe effects, is tricky. In particular, there are two main different distance definitions, both mimicking properties that distance has in Euclidean space: angular diameter distance and luminosity distance.

At first, notice that, exploiting the metric Eq. (1.5), we can derive the expression for the so-called comoving metric distance (which corresponds to the comoving distance for $\kappa = 0$) as

$$\eta = w = \int_t^{t_0} \frac{dt}{a} = \int_0^z \frac{dz}{H(z)}, \quad (1.17)$$

where we defined the cosmological redshift

$$1 + z := \frac{\lambda_o - \lambda_e}{\lambda_e} = \frac{1}{a}, \quad (1.18)$$

where $_o$ and $_e$ stand for “observed” and “emitted” respectively, and the last equality comes from the $p \propto 1/a$ behaviour of massless particles in FLRW.

Using Eq. (1.16) with $\Omega_r = \Omega_\kappa = 0$, we have

$$w(z) = \frac{1}{H_0 \Omega_{\Lambda,0}^{1/2}} \int_0^z \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} (1+z')^3 + 1 + \frac{\Omega_{\kappa,0}}{\Omega_{\Lambda,0}} (1+z')^2 \right)^{-1/2} dz'. \quad (1.19)$$

The comoving distance will then be $f_\kappa(w)$.

Angular diameter distance. The comoving distance is not a quantity we can actually measure. Hence, in cosmology one usually defines other types of distances. One example is the angular diameter distance. Consider an object with physical transverse length D ; then, in analogy with the Euclidean case, one can define the angular diameter distance, with $\delta\theta$ the angle subtended by the object, as

$$D_A := \frac{D}{\delta\theta}; \quad (1.20)$$

the FLRW line element implies the following relation for D

$$D = a(t) f_\kappa(w(t)) \delta\theta, \quad (1.21)$$

so that

$$D_A(z) = \frac{f_\kappa(w(z))}{1+z}. \quad (1.22)$$

In gravitational lensing, it will prove useful to have an expression of the angular diameter distance of a source at redshift z_2 as seen by an observer at redshift z_1 . To find such an expression, notice that the observer at z_1 would write

$$D'_A(z') = \frac{f_{\kappa'}(w(z'))}{1+z'}, \quad 1+z' = \frac{1+z_2}{1+z_1}, \quad \kappa' = \frac{\kappa}{a_1^2} = (1+z_1)^2 \kappa, \quad (1.23)$$

where z' is the redshift of the source as seen by the observer in z_1 and κ' is how observer at z_1 would normalize the curvature parameter according to the rescaling Eq. (1.8) with $\lambda = 1 + z_1$. Then, we have

$$w(z_2) - w(z_1) = \int_{z_1}^{z_2} \frac{dz}{H(z)} = (1+z_1) \int_0^{z'} \frac{dz''}{H((1+z'')(1+z_1)-1)} = (1+z_1) w', \quad (1.24)$$

where in the integral we changed the variable $1 + z = (1 + z'')(1 + z_1)$ and where in the last step we recognized the expected scaling of w according to Eq. (1.8). We can thus write

$$D'_A(z') =: D_A(z_2, z_1) = \frac{1 + z_1}{1 + z_2} f_{\kappa'} \left(\frac{w(z_2) - w(z_1)}{1 + z_1} \right) = \frac{f_{\kappa}(w(z_2) - w(z_1))}{1 + z_2}, \quad (1.25)$$

where the last step follows from the form of f_{κ} , Eq. (1.6), for any κ .

Luminosity distance. Very similarly, again in analogy with the Euclidean case, one can define the luminosity distance D_L via

$$F = \frac{L}{4\pi D_L^2}, \quad (1.26)$$

where F is the flux (energy per time per area) received from an observer, coming from a source with luminosity L (emitted energy per time in the emitter frame). To write a formula for D_L , one has to be careful about expansion of the universe effects. Schematically, the luminosity in the emitter frame is

$$L_e \sim \frac{\Delta E_e}{\Delta t_e}, \quad (1.27)$$

whereas, when we measure the flux, we see a different $\Delta E_o / \Delta t_o$. The energy redshifts as $\Delta E_o = \Delta E_e a(t_e)$; to understand what is Δt_o , suppose that a source located at comoving (not conformal) coordinates x emits a light pulse at t_e and at $t_e + \Delta t_e$; the comoving distance the two pulses travel to the observer are the same, hence

$$x = \int_{t_e}^{t_o} \frac{dt}{a} = \int_{t_e + \Delta t_e}^{t_o + \Delta t_o} \frac{dt}{a} \implies \int_{t_o}^{t_o + \Delta t_o} \frac{dt}{a} - \int_{t_e}^{t_e + \Delta t_e} \frac{dt}{a} = 0; \quad (1.28)$$

for small intervals of time, the previous indeed implies

$$\frac{\Delta t_o}{a(t_o)} = \frac{\Delta t_e}{a(t_e)}. \quad (1.29)$$

Moreover, at the time of observation, the light spread out to an area $A = 4\pi f_{\kappa}^2(w)$, ending up with

$$F = \frac{\Delta E_o}{4\pi f_{\kappa}^2(w) \Delta t_o} = \frac{a^2(t_e) L_e}{4\pi f_{\kappa}^2(w)} \implies D_L = f_{\kappa}(w)(1 + z). \quad (1.30)$$

1.1.3 Fluctuations and power spectra

In the previous sections, we described the equations pertaining an homogeneous and isotropic universe. We now need to go beyond this assumption if we want to describe the formation and growth of structures like clusters and galaxies. As we already remarked, in the early universe departures from homogeneity and isotropy can be characterized by the matter density fluctuation δ , defined as

$$\delta = \frac{\rho_m - \bar{\rho}_m}{\bar{\rho}_m}, \quad (1.31)$$

where ρ_m is the full matter density, comprehensive of the background mean density $\bar{\rho}_m$ (obeying Friedmann equations) and the fluctuation $\bar{\rho}_m \delta$.

We can view the density field as a stochastic field¹. The inflation paradigm predicts primordial fluctuations to be Gaussian, and a Gaussian random field is completely determined once we know its mean

¹We are forced to model our universe as a stochastic realization of a statistical ensemble of possibilities, since we do not have direct observational access to the primordial perturbations seeds (which will yield deterministic initial conditions); moreover, it

and the two-point correlation function. For δ , the mean is zero by definition, hence we only need the two-point correlator ξ_δ , defined as

$$\langle \delta(\vec{r}, t) \delta(\vec{r}', t) \rangle =: \xi_\delta(|\vec{r} - \vec{r}'|), \quad (1.32)$$

where the average is taken over all the possible stochastic realizations of δ . Due to homogeneity and isotropy, ξ depends only on the distance between the two points considered. Usually in cosmology one focuses on the power spectrum P_δ , that is defined, in Fourier space, as

$$\langle \delta(\vec{k}, t) \delta(\vec{k}', t) \rangle =: (2\pi)^3 \delta_D(\vec{k} + \vec{k}') P_\delta(k, t); \quad (1.33)$$

using this definition, it is easy to see that the power spectrum is the Fourier transform of ξ_δ (this fact is also known as the Wiener-Khinchine theorem [Wie30; Khi34]). We must remark that higher order (connected) correlators will become different from zero on later times even when dealing with initial Gaussian fluctuations, due to gravitational interactions which are non-linear. We will thus need also them to fully characterize the non-linearly evolved density stochastic field.

1.1.4 Fluctuations evolution

A full relativistic treatment of the fluctuations evolution must deal with subtleties like gauge choices [MW09]; for our purposes, since we will deal with non-relativistic CDM and scales smaller than the Hubble radius, the equations of motion reduce to the ones pertaining Newtonian gravity. The effects of the expansion of the universe are encoded only in the redefinition of the coordinates used (comoving coordinates are the proper ones) and in the redefinition of the gravitational potential. This is the Newtonian approach we will use in this section.

Here we will briefly go through the standard Eulerian perturbation theory, where one implements the Newtonian approach, by exploiting the so called Vlasov equation for collisionless dark matter particles (which give the most important contribution when dealing with LSS after radiation domination). A good reference for this topic is Ref. [Ber+02]. We assume we are in an era where only Ω_m and Ω_Λ are relevant, with a nearly flat space, $\Omega_m + \Omega_\Lambda = 1$. In practice, this is the setting relevant for times from deep in the matter domination era onwards.

Lagrangian of collisionless dark matter. We want here to derive an expression for the Lagrangian of collisionless dark matter particles in an expanding space-time. We will need it in order to obtain the Vlasov equation, the equation for the dark matter phase space in an expanding universe.

Consider at first the following Lagrangian for a single particle of mass m

$$L = \frac{1}{2} m \dot{\vec{r}}^2 - m \varphi(\vec{x}, t), \quad (1.34)$$

where here we will consider $\vec{x} = \vec{r}/a$ as the comoving distance, $\dot{\cdot} := d/dt$ and $\varphi(\vec{x}, t)$ as the Newtonian gravitational potential satisfying

$$\nabla_{\vec{x}}^2 \varphi(\vec{x}, t) = 4\pi G a^2 \rho(\vec{x}, t), \quad (1.35)$$

where we recall $\rho(\vec{x}, t) = \bar{\rho}_m(t)(1 + \delta(\vec{x}, t)) + \Lambda/(8\pi G)$. If we change the Lagrangian adding a total time derivative, we do not change the equations of motion; we can thus deal with the equivalent L'

$$L' = L - \frac{d}{dt} \left(\frac{1}{2} m a \dot{x}^2 \right), \quad (1.36)$$

is hopeless to study the evolution of a single, definite large scale structure due to the time scales involved. See e.g. Ref. [Ber+02] for more discussion on this point.

and the new Lagrangian becomes (we drop the prime)

$$L = \frac{1}{2}ma^2\dot{x}^2 - \frac{m}{2}a\ddot{x}x^2 - m\varphi = \frac{1}{2}ma^2\dot{x}^2 - m\Phi, \quad (1.37)$$

where in the last step, with $\Phi := \varphi - \varphi_b$, we exploited the second Friedmann equation Eq. (1.12)

$$\ddot{a} = -\frac{4\pi G}{3}a\bar{\rho}_m(t) + \frac{\Lambda}{3}; \quad (1.38)$$

and defined the background potential φ_b as

$$\varphi_b := \frac{2}{3}\bar{\rho}_m(t)\pi Ga^2x^2 - \frac{a^2}{6}\Lambda x^2. \quad (1.39)$$

With such φ_b definition, the gravitational potential Φ is sourced by the matter fluctuations alone, as φ_b relates to the background quantities, since it solves Eq. (1.35) with $\rho(\vec{x}, t) = \bar{\rho}_m(t) + \Lambda/(8\pi G)$. Notice that we can rewrite the Poisson equation Eq. (1.35), exploiting the definition of Ω_m and Eq. (1.39), as

$$\nabla_x^2 \Phi = \frac{3}{2}\Omega_m a^2 H^2 \delta. \quad (1.40)$$

Vlasov equation. We are interested in the collisionless Boltzmann equation; we will need the Hamiltonian

$$H = \vec{p} \cdot \vec{x} - L = \frac{p^2}{2ma^2} + m\Phi, \quad (1.41)$$

where we used

$$\vec{p} = \frac{\partial L}{\partial \dot{x}} = ma^2 \dot{x}. \quad (1.42)$$

The collisionless Boltzmann equation for the particle distribution function $f(\vec{x}, \vec{p}, t)$ is

$$\mathbb{C}[f] = 0 = \frac{\partial f}{\partial t} + \dot{x} \cdot \nabla_{\vec{x}} f + \dot{p} \cdot \frac{df}{d\vec{p}}, \quad (1.43)$$

and using the Hamilton equations

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{ma^2}, \\ \dot{p} &= -\frac{\partial H}{\partial \vec{x}} = -m\nabla\Phi, \end{aligned}$$

we find the Vlasov equation

$$\frac{\partial f}{\partial t} + \frac{\vec{p}}{ma^2} \cdot \nabla_{\vec{x}} f - m\nabla_{\vec{x}}\Phi \cdot \frac{df}{d\vec{p}} = 0. \quad (1.44)$$

Continuity and Euler equations. Solving the non linear and non local Vlasov equation Eq. (1.44) is hard; we can simplify the task by considering its first two moments:

$$\rho_m(\vec{x}, t) = \frac{m}{a^3} \int d^3p f(\vec{x}, \vec{p}, t), \quad (\text{zero moment})$$

$$\vec{v}(\vec{x}, t) = \frac{N}{ma} \int d^3p \vec{p} f(\vec{x}, \vec{p}, t) = \frac{1}{a^4 \rho_m} \int d^3p \vec{p} f(\vec{x}, \vec{p}, t), \quad (\text{first moment})$$

where in the first we divided by a^3 since we are interested in the physical volume and not the comoving one, and on the last $N = (\int d^3p f)^{-1} = m/a^3 \rho_m$ is a normalization factor.

Now taking Eq. (1.44), multiplying it by m and integrating over $\vec{p}$ we find

$$\dot{\rho}_m + 3H\rho_m + \frac{1}{a}\nabla_{\vec{x}} \cdot (\rho_m \vec{v}) = 0, \quad (1.45)$$

that is the continuity equation, whereas if again we multiply Eq. (1.44) by p^i and integrate over $\vec{p}$ we find

$$\frac{\partial v^i}{\partial t} + H v^i + \frac{1}{a} v^j \frac{\partial v^i}{\partial x^j} = -\frac{1}{a} \frac{\partial \Phi}{\partial x^i} - \frac{1}{a} \frac{\partial}{\partial x^j} (\Pi^{ij} \rho_m), \quad (1.46)$$

where at the end we have a quantity related to the second moment (that represents the dispersion of velocity)

$$\Pi^{ij} = \frac{\langle p^i p^j \rangle}{m^2 a^2} - v^i v^j, \quad (1.47)$$

where here the angle brackets means average over phase space coordinates. We see that an equation for the n -th moment contains also the $n + 1$ -th moment. To truncate this set of equations, we can assume for example that the Π^{ij} term is zero (i.e. negligible velocity dispersion, this is the so called single stream approximation), thus obtaining

$$\frac{\partial \vec{v}}{\partial t} + H \vec{v} + \frac{1}{a} (\vec{v} \cdot \nabla) \vec{v} = -\frac{1}{a} \nabla \Phi, \quad (1.48)$$

that is the Euler equation for a non static background.

Fluctuation equations. Expressing Eq. (1.35) with respect to the fluctuation δ , we obtain (exploiting the third Friedmann equation $\dot{\rho}_m = -3H\rho_m$ to get rid of the background density)

$$\dot{\delta} + \frac{1}{a} \nabla \cdot ((1 + \delta) \vec{v}) = 0; \quad (1.49)$$

we can take its time derivative and exploit Eq. (1.48) and Eq. (1.40) to obtain, at linear level on the fluctuations δ , v , Φ (notice that, at linear level, $\dot{\delta} = -a^{-1} \nabla \cdot \vec{v}$)

$$\ddot{\delta} + 2H\dot{\delta} - \frac{3}{2}\Omega_m H^2 \delta = 0. \quad (1.50)$$

This is the equation for fluctuations at linear level, valid as long as the fluctuations remain small. In Einstein-De Sitter models, $H = 2/3(1 + w_s)t$; with the ansatz $\delta \propto t^n$, we find

$$n = \begin{cases} 2/3, & \text{matter dominated,} \\ \pm 1, & \text{radiation dominated.} \end{cases} \quad (1.51)$$

We clearly see that we have a growing mode and a decaying mode, whose definition can be respectively read in

$$\delta(t) = \delta(t^{(i)}) \frac{D_{\pm}(t)}{D_{\pm}(t^{(i)})}. \quad (1.52)$$

The linearly evolved power spectrum, neglecting the decaying mode, will thus be $P_{\delta}^{\text{linear}} \propto D_+^2 P_{\delta}^{(i)}$, where D_+ here is the growing mode for CDM.

We can insert D_+ back in (1.50) and obtain

$$\ddot{D}_+ + 2H\dot{D}_+ - \frac{3}{2}H^2\Omega_m D_+ = 0; \quad (1.53)$$

a good fit formula for the solution of the previous for the standard Λ CDM model is found to be (see [Lah+91; CPT92; Car01])

$$D_+(a) = \frac{5a}{2} \Omega_m(a) \left(\Omega_m^{4/7}(a) - \Omega_\Lambda(a) + \left(1 + \frac{\Omega_m(a)}{2} \right) \left(1 + \frac{\Omega_\Lambda(a)}{70} \right) \right)^{-1}. \quad (1.54)$$

1.1.5 Lagrangian perturbation theory

In the previous section, we described the basics of Eulerian perturbation theory. For our purposes later on, especially for our discussion in Sec. 4.3, it will be useful to talk about Lagrangian perturbation theory as well. While in Eulerian perturbation theory one describes the evolution of the density field in terms of comoving coordinates $\vec{x}$, in Lagrangian perturbation theory one assigns coordinates $\vec{q}$ to all the particles considered and then one tracks on time the position of the particle labelled by $\vec{q}$. In equations, we have [Ber+02; VSB15; VWA15]

$$\vec{x}(\vec{q}, t) = \vec{q} + \vec{\psi}(\vec{q}, t), \quad (1.55)$$

where $\vec{\psi}$ is the so-called displacement field. Linear level perturbations on $\vec{\psi}$ contain non-linearities that linear Eulerian perturbation theory does not capture [Ber+02].

In Fourier space, the matter density fluctuations can be written, exploiting Eq. (1.55), as

$$\delta(\vec{k}) = \int d^3q e^{-i\vec{k}(\vec{q} + \vec{\psi})}. \quad (1.56)$$

We can thus write the two-point correlator as

$$\langle \delta(\vec{k}_1, z_1) \delta(\vec{k}_2, z_2) \rangle = \int d^3q_1 \int d^3q_2 \langle e^{-i\vec{k}_1(\vec{q}_1 + \vec{\psi}_1)} e^{-i\vec{k}_2(\vec{q}_2 + \vec{\psi}_2)} \rangle, \quad (1.57)$$

where we used the shorthand $\vec{\psi}_i := \vec{\psi}(\vec{k}_i, t_i)$; we can change variable $\vec{Q} = (\vec{q}_1 + \vec{q}_2)/2$, $\vec{q} = \vec{q}_2 - \vec{q}_1$; the Jacobian of this change of variables is one, and for isotropy the above correlator can depend only on $\vec{q}$; we can thus perform the integral over $\vec{Q}$ and end up with

$$\langle \delta(k_1, z_1) \delta(k_2, z_2) \rangle = (2\pi)^3 \delta_D(\vec{k}_1 + \vec{k}_2) \int d^3q e^{-i\vec{q} \cdot \vec{k}} \langle e^{-i\vec{k}(\vec{\psi}_2 - \vec{\psi}_1)} \rangle; \quad (1.58)$$

this translates into the following formula for the power spectrum

$$P_\delta(k, z_1, z_2) = \int d^3q e^{-i\vec{q} \cdot \vec{k}} \langle e^{-i\vec{k}(\vec{\psi}_2 - \vec{\psi}_1)} \rangle. \quad (1.59)$$

The cumulant theorem tells us

$$\langle e^{-i\vec{k}(\vec{\psi}_2 - \vec{\psi}_1)} \rangle = \exp \left(\sum_{n>0} \frac{(-i)^n}{n!} \langle (\vec{k} \cdot (\vec{\psi}_2 - \vec{\psi}_1))^n \rangle_c \right), \quad (1.60)$$

where $_c$ stands for connected. For isotropy, the $n = 1$ term is zero; for $n = 2$, we have to deal with terms like $k_i k_j \langle \psi_l^i \psi_m^j \rangle$, $l, m = 1, 2$; in particular, we have

$$\langle \psi_l^i \psi_l^j \rangle = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} e^{i\vec{q}_l(\vec{k}_1 + \vec{k}_2)} \underbrace{\langle \psi_l^i(k_1) \psi_l^j(k_2) \rangle}_{(2\pi)^3 \delta_D(\vec{k}_1 + \vec{k}_2) P_\psi^{ij}(k)} = \int \frac{d^3k}{(2\pi)^3} P_\psi^{ij}(k, z_l), \quad (1.61)$$

and for isotropy (notice that the dependence on q disappears here), we can define

$$\langle \psi_l^i \psi_l^j \rangle =: \delta_{ij} \Sigma^2(z_l), \quad \Sigma^2(z_l) = \frac{1}{3} \int \frac{d^3 k}{(2\pi)^3} \delta_{mn} P_\psi^{mn}(k). \quad (1.62)$$

We can thus write the power spectrum as

$$P_\delta(k, z_1, z_2) = e^{-k^2(\Sigma^2(z_2) + \Sigma^2(z_1))/2} \int d^3 q e^{-i\vec{q} \cdot \vec{k}} e^{k_i k_j \langle \psi_2^i \psi_1^j \rangle} (1 + \text{higher orders}). \quad (1.63)$$

For $n > 2$, we have terms which are non-zero only when non-linearities are taken into account (recall that you should consider only connected correlators in the exponential), so what we call higher order terms are non-linear corrections arising for $n > 2$ cumulants.

In linear theory (Zel'dovich approximation), we can write [Mat08; Ber+02]

$$\psi^i(k, z) = i k^i D_+(z) \delta(t_0, k) / k^2, \quad (1.64)$$

so that we can write the linear part of $\Sigma(z)$, which we denote with $\Sigma_L(z)$, as

$$\Sigma_L^2(z) = \frac{D_+^2(z)}{3} \int_0^\infty \frac{dk}{2\pi^2} P_L(k, z=0) =: D_+^2(z) \Sigma_0^2, \quad (1.65)$$

where $D_+(z)$ is the growth factor normalized to unity at $z=0$, which obeys $\delta(k, z) = D_+(z) \delta(k, z=0)$, and $P_L(k, z=0)$ is the linearly evolved power spectrum until today, i.e. the power spectrum related to the density field fluctuations evolved as in Eq. (1.52). In linear theory we thus have (rearranging the $\exp(-k^2(D_+^2(z_2) + D_+^2(z_1))\Sigma_0^2/2)$ term)

$$P_L(k, z_1, z_2) = e^{-k^2((D_+(z_2) - D_+(z_1))^2 + 2D_+(z_1)D_+(z_2))\Sigma_0^2/2} \int d^3 q e^{-i\vec{q} \cdot \vec{k}} e^{k_i k_j D_+(z_1)D_+(z_2)\eta_{ij}^0}, \quad (1.66)$$

where we defined $\eta_{ij} := \langle \psi_2^i \psi_1^j \rangle$ and η_{ij}^0 analogously to Σ_0 in Eq. (1.65). Notice that we can rewrite the previous as

$$P_L(k, z_1, z_2) = e^{-k^2(D_+(z_2) - D_+(z_1))^2 \Sigma_0^2/2} P_L(k, \bar{z}), \quad (1.67)$$

where $D_+(z_1)D_+(z_2) =: D_+^2(\bar{z})$, with $\bar{z}$ an appropriate mean redshift which can be determined using the form of the linear growth factor D_+ . Eq. (1.67) will suggest an ansatz for the non-linear non-equal time correlator, to be discussed in next section. Don't be confused by the notation, here P_L is the linear power spectrum in the Lagrangian picture, meaning it is the power spectrum in the Zel'dovich approximation. In particular, $P_L(k, z) \neq D_+^2(z) P_\delta(k, z=0)$, differently to what happens in the linear power spectrum in the Euclidean picture.

For reference, the explicit expression of η_{ij}^0 is

$$\eta_{ij}^0 = -\frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \int \frac{dk}{2\pi^2} \frac{j_0(kq)}{k^2} P_L^0(k) = \int \frac{dk}{2\pi^2} P_L^0(k) \left(\left(j_0(kq) - \frac{1}{3} j_2(kq) \right) \delta_{ij} + j_2(kq) \hat{q}_i \hat{q}_j \right), \quad (1.68)$$

where we exploited

$$\frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} j_0(kq) = k^2 \left(\left(\frac{1}{3} j_2(kq) - j_0(kq) \right) \delta_{ij} - j_2(kq) \hat{q}_i \hat{q}_j \right). \quad (1.69)$$

1.1.6 Non-equal time correlator approximation

Expressions for the variance of projection fields, like the external convergence which will be discussed in Sec. 2.4, need a reliable estimate of the non-equal time matter power spectrum $P_\delta(k, z, z')$, Eq. (1.63)². This problem has been extensively studied in the literature, both analytically and numerically (see for instance [KKS97; HS03; DW04; Mom+06; Suy+10] and references therein). We want here to provide a simple, yet accurate enough analytical approximation to $P_\delta(k, z, z')$. What we show in this section is a novel result of this thesis, which appeared in Ref. [Teo+22].

In linear theory, the non-equal time matter power spectrum is simply given by

$$P_\delta(k, z, z') = D_+(z)D_+(z')P_L(k) ; \quad (1.70)$$

however, since a significant contribution to projection fields may come from very non-linear scales, the linear theory estimate is not reliable, as we will show in Sec. 4.3 for the external convergence.

To get a more reliable theoretical estimate, one has to use the non-linear matter power spectrum, which can be simply obtained using HALOFIT [PS14; Tak+12]. Unfortunately, HALOFIT outputs the non-linear power spectrum only at equal times. To extend this output to non-equal times requires some approximations. Inspired by the linear theory, the commonly used prescription is

$$P_\delta(k, z, z') = \sqrt{P_\delta(k, z)P_\delta(k, z')} . \quad (1.71)$$

We are going to argue that this and other similar approximations do not properly capture the non-equal time matter power spectrum on small scales. The reason is large bulk flows, which displace the dark matter particles by $\mathcal{O}(10)$ Mpc. These large displacements exactly cancel for equal time correlation functions³, due to the equivalence principle. However, for non-equal time correlation functions, the dark matter particles are displaced by different amounts, depending on times at which the density fields are evaluated. On scales smaller than $\mathcal{O}(10)$ Mpc, this leads to exponential suppression of power in the non-equal time power spectrum. This important effect is not captured by Eq. (1.71).

One can show that the same exponential suppression of Eq. (1.66) remains going to higher orders in perturbation theory. However, beyond Zel'dovich approximation, the non-linear spectra cannot be simply expressed through the equal time counterparts anymore. The general structure of the one-loop result (i.e. the next to leading order result) can be recovered via the following steps. From Eq.(1.63), we can isolate all the non linear term; for example, with $\Delta\Sigma_i^2 := \Sigma^2(z_i) - D^2(z_i)\Sigma_0^2$, we can rewrite

$$-\frac{1}{2}(\Sigma^2(z_2) + \Sigma^2(z_1)) = -\frac{1}{2}\Sigma_0^2(D_+(z_2) - D_+(z_1))^2 - D_+(z_2)D_+(z_1)\Sigma_0^2 - \frac{1}{2}(\Delta\Sigma_2^2 + \Delta\Sigma_1^2) ; \quad (1.72)$$

we end up, defining also $\Delta\eta_{ij} := \eta_{ij} - D_+(z_1)D_+(z_2)\eta_{ij}^0$, with

$$P_\delta(k, z_1, z_2) = e^{-k^2\Sigma_0^2(D_+(z_2)-D_+(z_1))^2/2 - k^2D_+(z_1)D_+(z_2)\Sigma_0^2} \int d^3q e^{-i\vec{k}\cdot\vec{q}} e^{k_i k_j D_+(z_1)D_+(z_2)\eta_{ij}^0} \\ \times e^{-k_i k_j (\delta_{ij}\Delta\Sigma_1^2 + \delta_{ij}\Delta\Sigma_2^2 - 2\Delta\eta_{ij})/2} (1 + \text{higher orders}) . \quad (1.73)$$

This implies the generic structure

$$P_{1\text{-loop}}(k, z_1, z_2) = e^{-k^2\Sigma_0^2(D_+(z_2)-D_+(z_1))^2/2} [P_{1\text{-loop}}(k, \bar{z}) + \delta P(k, z_1, z_2)] , \quad (1.74)$$

²We note that simply neglecting the non-equal time contribution to the correlator can bring biases when discussing projection fields [KH17].

³Large displacements can have observable effects only for sharp features in the correlation functions. BAO peak is one such feature and large displacements lead to the spread of the BAO peak, or damping of the BAO wiggles in the power spectrum.

where δP contains $\Delta\Sigma_i^2$, $\Delta\eta_{ij}$ terms, and we call 1-loop the higher order correction to the linear power spectrum Eq. (1.67). The exact form of $\delta P(k, z_1, z_2)$ is not important, but we know that it has two important properties. First, this correction is small in perturbation theory [VCW16; Sch+19]. Second, $\delta P(k, z_1, z_2)$ vanishes for equal times. Therefore, we expect that the correction to the equal-time one-loop term in the square brackets is always small. Furthermore, given the expectation that $P(k, z_1, z_2)$ is a smooth function of z_1 and z_2 , when the two times are not equal, the exponential suppression at high k is always large enough to make any small mistake in the modelling of the non-linear power spectrum insignificant. Neglecting the term $\delta P(k, z_1, z_2)$ translates into the approximation

$$P_\delta(k, z_1, z_2) = e^{-k^2 \Sigma_0^2 (D_+(z_2) - D_+(z_1))^2 / 2} P(k, \bar{z}), \quad (1.75)$$

which will be the equation that we will use to compute non-equal time correlators. Notice that the previous is exact at the linear level, as we showed in Eq. (1.67). Such formulation will be relevant for the results we show in Sec. 4.3.

1.2 Dark matter from star kinematics

From CMB and LSS, within the Λ CDM model outlined in Sec. 1.1, one can infer cosmological parameters like H_0 and $\Omega_{m,0}$, the dark matter abundance today. The fact that both CMB and LSS give the DM contribution to Ω_m to be much different from zero (in fact, around 5 times the contribution from baryons) gives one of the strongest evidence for the existence of DM.

However, DM evidence was originally discovered via its influence on stellar kinematics, out of which one can infer the presence of much more mass than the one that can be accounted for with visible matter. In this section, we briefly outline how to connect stellar kinematics with mass measurements of galactic systems. Such probes of DM will be relevant for our discussion in Ch. 6.

In Sec. 1.2.1 and Sec. 1.2.2, we briefly review the concepts of circular velocity in spiral galaxies and velocity dispersion, the latter relevant for dwarf galaxies, elliptical galaxy and galaxy clusters.

1.2.1 Circular velocity in spiral galaxies

In a spiral galaxy, stars move in a circular orbit. The circular velocity of stars $v_{\text{circ}}(r)$ as a function of radius r can be connected to the total mass (including stars, gas, DM) $M(r)$ inside the radius r as

$$v_{\text{circ}} = \sqrt{\frac{GM(r)}{r}}. \quad (1.76)$$

The rotation curve of galaxies at large radii is close to flat, implying $M(r) \sim r$ which implies $\rho(r) \sim r^{-2}$. The density of luminous matter does not have that behaviour on the outskirts of galaxies, suggesting the presence of a dark component. Interestingly, cosmological simulations of collisionless CDM predict dark matter halos to have a density profile which is reasonably fit by the functional form

$$\rho(r) = \frac{\rho_s R_s^3}{r(r + R_s)^2}, \quad (1.77)$$

with ρ_s and R_s parameters of the profile. This is called the Navarro-Frenk-White (NFW) profile [NFW97]. Such a profile is consistent (or better, not inconsistent) with the claimed $\rho \sim r^{-2}$, valid for certain ranges of $r/R_s \sim 1$.

1.2.2 Stellar velocity dispersion in galactic systems

Not every galactic system is rotationally supported. Stars in dwarf galaxies and elliptical galaxies, or even galaxies themselves in groups/clusters, are better described via the Line Of Sight Velocity Dispersion (LOSVD). In the following, we review how to connect the observable LOSVD with the parameters of the galaxy/cluster, like its mass profile.

Stars within a DM halo or galaxies within a NFW group/cluster can be described by the phase-space function $f(\vec{x}, \vec{v})$, introduced in Sec. 1.1.4. Building on the results of Eq. (1.44) specialized to a stationary distribution function $f(\vec{x}, \vec{v})$ in flat space, we have

$$\vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0. \quad (1.78)$$

One can obtain the number density of tracers n_* (here tracers can mean stars or galaxies) and their velocity dispersion $\vec{\sigma}$ via the zero and first moment of f ,

$$n_* = \int f d^3v, \quad \vec{\sigma} = \int d^3v \vec{v} f; \quad (1.79)$$

by multiplying Eq. (1.78) with $\int d^3v$, one can obtain, in spherical coordinates and assuming zero higher momenta [Cap20]

$$\frac{\partial(n_*\sigma_r^2)}{\partial r} + n_* \frac{(1+\beta)\sigma_r^2 - \sigma_\phi^2}{r} = -n_* \frac{\partial \Phi}{\partial r}, \quad (1.80)$$

$$(1-\beta) \frac{\partial(n_*\sigma_r^2)}{\partial \theta} + n_* \frac{(1-\beta)\sigma_r^2 - \sigma_\phi^2}{\tan \theta} = -n_* \frac{\partial \Phi}{\partial \theta}, \quad (1.81)$$

where $\beta(r)$ is the anisotropy parameter, defined as

$$\beta(r) = 1 - \frac{\sigma_\theta^2(r) + \sigma_\phi^2(r)}{2\sigma_r^2(r)}; \quad (1.82)$$

if one assumes $\sigma_\phi^2 = \sigma_\theta^2$, one can simplify Eq. (1.80) as

$$\frac{\partial(n_*\sigma_r^2)}{\partial r} + 2\beta n_* \frac{\sigma_r^2}{r} = -n_* \frac{\partial \Phi}{\partial r}, \quad (1.83)$$

which is the spherical Jeans equation, whose solution reads

$$n_*(r)\sigma_r^2(r) = G \int_r^\infty ds \frac{n_*(s)M(s)}{s^2} e^{2 \int_r^s dz \beta(z)/z}, \quad (1.84)$$

where $M(s)$ is the total mass enclosed in the radius s , connected to the gravitational potential via

$$\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r}. \quad (1.85)$$

A popular parametrization for $\beta(r)$ is the ellipsoidal Osipkov-Merritt anisotropy formulation ([Osi79; MA85]), for which

$$\beta(r) = \frac{r^2}{r^2 + r_a^2}, \quad (1.86)$$

with r_a the anisotropy radius parameter.

The observable is the line of sight velocity dispersion σ_{los} ; averaged over the tracers n_* , denoting Σ_* as the projected n_* profile, we can write

$$\Sigma_* \sigma_{\text{los}}^2 = \int_{-\infty}^{\infty} n_* \sigma_z^2 dz ; \quad (1.87)$$

since

$$\frac{\partial}{\partial z} = \cos \theta \frac{\partial}{\partial r} - \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \implies v_z = \cos \theta v_r - \sin \theta v_\theta , \quad (1.88)$$

and with our assumption that higher orders do not contribute, we have

$$\sigma_z^2 = \int d^3v f v_z^2 = \cos^2 \theta \sigma_r^2 + \sin^2 \theta \sigma_\theta^2 = \left(1 - \frac{R^2}{r^2}\right) \sigma_r^2 + \frac{R^2}{r^2} \sigma_\theta^2 = \left(1 - \frac{R^2}{r^2} \beta\right) \sigma_r^2 , \quad (1.89)$$

where we defined $R^2 := x^2 + y^2$. While integrating over z , R remains constant. This means that we can write

$$z^2 = r^2 - R^2 \implies 2z dz = 2r dr \implies dz = \frac{r dr}{\sqrt{r^2 - R^2}} ; \quad (1.90)$$

this is for $r > R$ for $z > 0$; for $z < 0$, if the integrand is symmetric, one can write (this corresponds to an Abel transform of a function)

$$\int_{-\infty}^{\infty} dz f(z, R) = 2 \int_R^{\infty} \frac{r dr}{\sqrt{r^2 - R^2}} f(\sqrt{r^2 - R^2}, R) ; \quad (1.91)$$

finally,

$$\sigma_{\text{los}}(R) = \frac{2}{\Sigma_*} \int_R^{\infty} \frac{r dr}{\sqrt{r^2 - R^2}} \left(1 - \frac{R^2}{r^2} \beta\right) n_* \sigma_r^2 , \quad (1.92)$$

which, assuming $\beta(r)$ to follow a Osipkov-Merritt profile Eq. (1.86), and converting to angular variables via $R = D_1 \theta$, where D_1 is the galaxy angular diameter distance, can be rewritten as [ML05]

$$\sigma_{\text{los}}^2(\theta) = \frac{2G}{\Sigma_*(\theta)} \int_1^{\infty} \frac{dy}{y} K(y) \rho_*(y D_1 \theta) M(y D_1 \theta) , \quad (1.93)$$

with

$$K(u) = \frac{u_a^2 + 1/2}{(u_a^2 + 1)^{3/2}} \left(\frac{u^2 + u_a^2}{u} \tan^{-1} \sqrt{\frac{u^2 - 1}{u_a^2 + 1}} \right) - \frac{1}{2(u_a^2 + 1)} \sqrt{1 - \frac{1}{u^2}} . \quad (1.94)$$

For analytical estimates, we note that the isotropic velocity limit gives $K(u, \infty) = \sqrt{1 - 1/u^2}$.

If one averages over N data points in a field of view $\mathcal{A}$, i.e.

$$\sigma_{\text{obs}}^2 \sim \frac{1}{N} \sum_{i \in \mathcal{A}}^N \sigma_{\text{obs},i}^2 , \quad (1.95)$$

then the proper quantity to compare with the observed dispersion velocity is (in the “continuum limit”)

$$\langle \sigma_{\text{los}}^2 \rangle = \frac{\int_{\mathcal{A}} d^2x \Sigma_* \sigma_{\text{los}}^2}{\int_{\mathcal{A}} d^2x \Sigma_*} , \quad (1.96)$$

where $\mathcal{A}$ is the aperture corresponding to the observation. Given a model for the mass profile $M(r)$ and for the anisotropy profile $\beta(r)$, one can obtain the parameters of such models by comparing with the observable σ_{los} .

1.3 Brief overview of the H_0 tension

As we remarked in the introduction, the Λ CDM model is most likely just an effective description of an underlying, more fundamental theory. To have hints of such a theory, there has been a lot of efforts into making many precision measurements with many different probes. As a result, some tensions on the measurement of cosmological parameters emerged, most notably on the measurement of H_0 between so-called early universe and late universe probes. Another H_0 probe, independent of the aforementioned ones, comes from gravitational lensing, which will be a major focus of this thesis. We here provide a brief overview of the H_0 tension, mentioning only the most important measurements plus the ones related to gravitational lensing (main focus of the thesis), without aiming to be exhaustive. For reviews, see e.g. [DV+21; VSGM23; DV+25].

CMB and LSS based measurements. The CMB temperature and polarization anisotropies can be fitted to the 6-parameter Λ CDM model. Results from Planck [Agh+20] give the less than 1% precise measurement of the Hubble constant, $H_0 = 67.36 \pm 0.54$ km/s/Mpc.

The study of LSS can also yield precise H_0 measurements [Abb+18; ISZ20; D'A+20; Tr20]. In particular, study of the BAO peak, when combined with different priors, yields $H_0 \sim 68$ km/s/Mpc [Ala+21], compatible with the Planck result.

CMB/LSS measurements are considered “early Universe” probes because they hinge crucially on high-redshift ($z \sim 10^3$) dynamics such as the baryonic perturbations sound horizon.

Distance ladder based method. A direct way of measuring H_0 would be to connect the luminosity distance D_L and measured redshift z to (relatively) far-away objects via Eq. (1.30). Unfortunately, direct estimates of D_L are hard to come by.

For objects in the Milky Way, one can measure distances via the parallax method. This method however cannot be applied for objects at extragalactic distance, which are needed to measure H_0 via this method. To overcome this problem, a possibility is to use (supposed) intrinsic properties, e.g. absolute luminosity, of a class of astrophysical objects within the Milky Way, whose distance can be measured with parallax. Once such property is calibrated, one can look for the same object in extragalactic distance and recover its true distance. This method is called distance ladder, as it relies on an H_0 measurement coming from different steps. The most famous distance ladder measurement comes from the SH0ES collaboration, which relies on Cepheids calibration as the first step of the ladder, to then calibrate SNIa luminosity curve with Cepheids as second step of the ladder. The latter step of the ladder is to use far-away SNIa luminosity curves to directly measure H_0 . Measurements within this method give $H_0 \simeq 73$ km/s/Mpc, with around $\lesssim 2\%$ uncertainty [Rie+22; Rie+19; Rie+24]. A main concern in the SNIa analyses is the calibration of local distance ladder anchors. The TRGB-calibrated SNIa analysis from the Carnegie Chicago Hubble project CCHP [Fre+19; FM23; Fre+24], which uses TRGB as the first step of the ladder, gives a somewhat lower H_0 value, $H_0 \sim 70$ km/s/Mpc with a $\sim 4\%$ uncertainty.

Distance ladder methods are considered “late Universe” probes of H_0 , that is, they involve only low-redshift ($z \sim 1$) dynamics, in contrast to the CMB/LSS result. The fact that these probes obtain different values may hint at a possible breakdown of the Λ CDM model. Due to this possibility, it is important to validate this result with other, independent, late universe H_0 probes, like the gravitational lensing one we discuss next.

Strong gravitational lensing H_0 measurement. Gravitational lensing offers another way of measuring H_0 via time-delays, as we will see in Sec. 2.2. The TDCOSMO collaboration uses massive elliptical galaxies with measured time-delays to infer H_0 . The first result of TDCOSMO, which used data from 7 gravitational lens systems, was $H_0 = 74.0^{+1.7}_{-1.8}$ km/s/Mpc [Mil+20]. This result is in tension with early

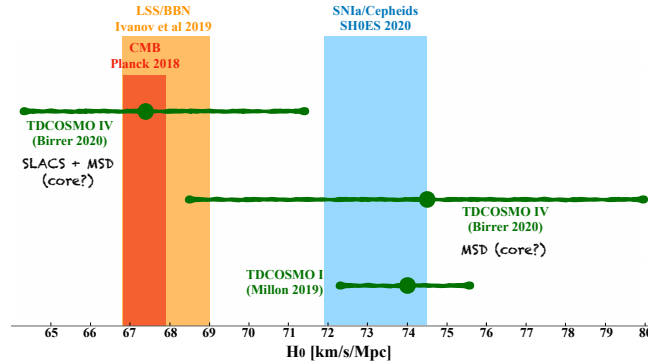


FIGURE 1.1: Status of the lensing H_0 measurements. The internal MSD, added in moving from Ref. [Mil+20] to Ref. [Bir+20], increased the error bars on the H_0 measurement.

universe probes. However, it is compatible with (and independent of) the SH0ES result, and both are late universe probes. This means that the result of Ref. [Mil+20] reinforces the tension.

However, modelling degeneracies, out of which MSD is a primary one, couple the inferred value of H_0 to the assumed density profile of the lens galaxy [FGS85; SS13a; SS14; Xu+16; BAR16; USS17; Tag+18; Son18; GW19; Koc20; Koc02a]. Ref. [BCS20] pointed out that a core component in the lens galaxy density profile could comprise an approximate MSD, shifting the inferred value of H_0 without affecting the image reconstruction. Subsequently, TDCOSMO-IV [Bir+20] added an effective “internal MSD” degree of freedom to their halo model fit; as a result, the error budget on H_0 increased to the level expected from stellar kinematics, around 10% [Koc20; Koc02a]. Interestingly, including galaxies from the Sloan Lens ACS (SLACS) survey [Bol+06] in the kinematics analysis, and making the additional assumption that SLACS and TDCOSMO galaxies share a self-similar structure, shifted the central value of the lensing H_0 to the CMB value while providing some positive evidence for an internal MSD component in the data. The status of the lensing H_0 measurements from TDCOSMO is illustrated in Fig. 1.1.

Subsequently, the TDCOSMO collaboration presented a forecast in Ref. [BT21a], suggesting that with future data—most notably resolved stellar kinematics—it will be possible to measure H_0 from gravitational lensing with a precision of 2%, even accounting for the potential effects of internal MSD. If this forecast proves accurate, future gravitational lensing measurements of H_0 could either strengthen the case for or challenge the existence of the H_0 tension.

The study of the MSD degeneracy in the strong gravitational lensing measurement of H_0 is a main part of the work of this thesis, and we will extensively return back to this point in Ch. 4.

Other H_0 measurements from gravitational lensing come from galaxy clusters [Gri+18; Gri+24]. Given the uncertainties on the cluster modelling, such measurements precision is order 10% as the current TDCOSMO measurement [Bir+20], making it compatible with both CMB/LSS and SH0ES.

Chapter 2

Gravitational Lensing

According to Einstein's general relativity, light follows null geodesics in spacetime. In the presence of massive bodies, spacetime curvature causes light rays to bend toward the massive body responsible for the distortion. This phenomenon, known as gravitational lensing, has several observable consequences, such as the formation of multiple images of the same source, distortions of images, and magnifications or demagnifications of light curves.

Depending on the magnitude of the spacetime curvature and the geometry of the gravitational lensing system, gravitational lensing can be classified into three main types:

- *Strong lensing.* When the deflection of light is on the order of arcseconds, noticeable effects such as multiple images, arcs, and Einstein rings can be observed (with sufficiently powerful telescopes). Strong lensing typically requires a very massive lens, with mass $M \sim 10^{13} - 10^{15} M_{\odot}$ (e.g., galaxy clusters or massive galaxies), and cosmological distances on the order of Gpc. Common sources for strong lensing include distant quasars and supernovae.
- *Weak lensing.* When the deflection is much smaller, the effects of gravitational lensing can still be detected statistically by observing coherent distortions and deformations of background sources. Weak lensing is a powerful tool for studying the distribution of DM and LSS in the universe.
- *Microlensing.* Even when light deflection and image distortion are too small to be directly observed, gravitational lensing can still produce measurable, time-dependent magnifications of background objects (if amplification is strong enough). A typical microlensing system involves a source star in another galaxy and a lensing star within the Milky Way.

Most of the material of this thesis will focus on strong gravitational lensing; we will touch on aspects of weak lensing in Sec. 2.4 and Sec. 4.3.

In the following, we review the basics of gravitational lensing, propaedeutic for understanding the main results presented in this thesis. In Sec. 2.1, we derive the lens equation from the geodesic deviation equation in the perturbed FLRW perturbed metric in Newtonian gauge; in Sec. 2.2, we show the basic consequences of the lens equation, together with discussion of time delays and the MSD; in Sec. 2.3, we extend the lens equation to the multiple sources case, together with LSS effects accounted in the tidal approximation; in Sec. 2.4, we discuss the external convergence expectation values, building on the results of Sec. 1.1.6.

2.1 The lens equation derivation

Consider a bundle of neighbouring light rays $x^{\mu}(\lambda, \vec{b})$, where $\vec{b}$ labels the ray ($\vec{b}$ could be a vector of three parameters representing for example the two angles at which the light-ray is observed and its phase, see Ref. [SEF92]) and λ is the affine parameter along that ray. The tangent vector to a light ray labelled

by $\vec{b}$ is

$$k^\mu = \frac{dx^\mu}{d\lambda} . \quad (2.1)$$

We call the displacement vector of neighbouring geodesics along a direction $\vec{b}$ as

$$\xi^\mu = b^i \frac{\partial x^\mu}{\partial b^i} . \quad (2.2)$$

The ray bundle $x^\mu(\lambda, \vec{b})$ should be continuous and differentiable in both λ and $\vec{b}$; in particular, the geodesic deviation vector should obey

$$\frac{D}{D\lambda} \xi^\mu = \xi^\nu \nabla_\nu k^\mu , \quad \frac{D}{D\lambda} := k^\mu \nabla_\mu , \quad (2.3)$$

where ∇_μ is the covariant derivative. This condition is a sort of generalization of the fact that second partial derivatives should commute, schematically $\partial/\partial b \partial x/\partial \lambda = \partial/\partial \lambda \partial x/\partial b$; the geodesic deviation equation reads

$$\frac{D^2 \xi^\mu}{D\lambda^2} = k^\rho k^\sigma \nabla_\rho \nabla_\sigma \xi^\mu = R^\mu_{\rho\sigma\nu} k^\sigma k^\rho \xi^\nu , \quad (2.4)$$

and we exploited $k^\mu \nabla_\mu k^\nu = 0$ (which is the geodesic equation). Using the Sachs formalism [JES61; Sac62], given an observer u^μ along the light ray bundle, we can define screen vectors s_A^μ , $A = 1, 2$, which project on the screen perpendicular to both u^μ and k^μ , which satisfy

$$s_A^\mu u_\mu = s_A^\mu k_\mu = 0 , \quad g_{\mu\nu} s_A^\mu s_B^\nu = \delta_{AB} ; \quad (2.5)$$

defining $\tilde{\xi}_A := \xi_\mu s_A^\mu$, with the parallel transport condition $k^\nu \nabla_\nu s_A^\mu = 0$ for the screen vectors, the geodesic deviation equation becomes

$$\frac{d^2 \tilde{\xi}_A}{d\lambda^2} = R_{AB} \tilde{\xi}^B . \quad (2.6)$$

We can split R_{AB} into a Ricci-term and a Weyl-term as in

$$R = \begin{pmatrix} \Phi_{\text{Ricci}} & 0 \\ 0 & \Phi_{\text{Ricci}} \end{pmatrix} + \begin{pmatrix} -\text{Re } \Psi_{\text{Weyl}} & \text{Im } \Psi_{\text{Weyl}} \\ \text{Im } \Psi_{\text{Weyl}} & \text{Re } \Psi_{\text{Weyl}} \end{pmatrix} , \quad (2.7)$$

where

$$\Phi_{\text{Ricci}} = -\frac{1}{2} R_{\mu\nu} k^\mu k^\nu , \quad \Psi_{\text{Weyl}} = -\frac{1}{2} (s_1^\mu - i s_2^\mu) (s_1^\sigma - i s_2^\sigma) C_{\mu\nu\rho\sigma} k^\nu k^\rho , \quad (2.8)$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor (see Eq. (C.6) for definition). The approximation which is usually employed is to split the cosmological contribution to R_{AB} and the deflector contribution, to be treated respectively with the FLRW metric Eq. (1.5) and the flat Newtonian metric

$$ds^2 = -dt^2 (1 + 2\Phi) + (1 - 2\Phi) dx^i dx^i . \quad (2.9)$$

In the following, we stick with this approximation. In App. B, we see how such approximation can be seen as a leading order expansion in the simple example of the Kottler metric, which is the simplest exact metric encompassing both expansion of the Universe and the presence of a point mass.

FLRW part. Using the FLRW metric, we have that the Weyl tensor $C_{\mu\nu\rho\sigma}$ is zero; using Einstein equations for a perfect fluid, together with $k^2 = 0$, we have from Eq. (2.8)

$$\Phi_{\text{Ricci}} = -4\pi G T_{\mu\nu} k^\mu k^\nu = -4\pi G \omega^2 \sum_i (\rho_i + P_i) , \quad (2.10)$$

where $\omega := -u^\mu k_\mu$ is the frequency of the light ray as seen by the observer u^μ . For a universe with just matter and a cosmological constant, the previous reads

$$\Phi_{\text{Ricci}} = -4\pi G \rho_m \omega^2 = -4\pi G \rho_{m,0} (1+z)^5, \quad (2.11)$$

where on the last step we exploited the result $\rho_m = \rho_{m,0} (1+z)^3$ and, via appropriate choice of the affine parameter for the light ray, we can have

$$-k_\mu u^\mu = \frac{1}{a} = 1+z. \quad (2.12)$$

The geodesic equation then reads

$$\frac{d^2 \xi_A}{d\lambda^2} = -4\pi G \rho_{m,0} (1+z)^5 \xi^A. \quad (2.13)$$

With the affine parameter normalization Eq. (2.12), since for a comoving observer $u^\mu = (1, 0, 0, 0)$, we have

$$-k^\mu u_\mu = k^0 = \frac{dt}{d\lambda} = \frac{1}{a} \implies dt = \frac{d\lambda}{a}; \quad dt = a d\eta = a d\omega \implies d\lambda = a^2 d\omega. \quad (2.14)$$

So (dots indicate derivatives with respect to t)

$$\frac{d^2}{d\omega^2} \frac{\xi^a}{a} = \frac{1}{a} \frac{d^2 \xi^a}{d\omega^2} - \xi^a a \dot{H} = -\kappa \frac{\xi^a}{a}, \quad (2.15)$$

where on the last step we exploited the Friedmann equations,

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 = -4\pi G \rho_0 (1+z)^3 + \frac{\kappa}{a^2}, \quad (2.16)$$

together with

$$\frac{d^2 \xi^a}{d\omega^2} = a^2 \frac{d^2 \xi^a}{d\lambda^2} = -4\pi G \rho_0 (1+z)^3 \xi^a. \quad (2.17)$$

We thus end up with

$$\left(\frac{d^2}{d\omega^2} + \kappa \right) \frac{\xi_A}{a} = 0. \quad (2.18)$$

Perturbation contribution. From the metric Eq. (2.9), which has non-zero Ψ_{Weyl} , one, with similar procedure as before, finds [Bar10]

$$\frac{d^2}{d\omega^2} \frac{\xi_A}{a} = -2\partial^i \Phi. \quad (2.19)$$

The full geodesic deviation equation. The equation for the geodesic displacement, defining $\xi^i = a f_\kappa(w) \beta^i$, $i = 1, 2$, reads

$$\left(\frac{d^2}{d\omega^2} + \kappa \right) f_\kappa(w) \beta^i = -2\partial^i \Phi, \quad (2.20)$$

where β^i will be the displacement angle of the light ray. The Green function of the previous is

$$G(w, w') = f_\kappa(w - w') \Theta(w - w'), \quad (2.21)$$

where Θ is the Heaviside function. We impose the initial conditions

$$f_\kappa(w=0) \beta^i(w=0) = 0; \quad \left. \frac{df_\kappa(w) \beta^i}{d\omega} \right|_{w=0} = \theta^i, \quad (2.22)$$

so that at the observer position, the rays coincide and arrive at an angle $\vec{\theta}$. Then,

$$f_\kappa(w)\beta^i(w, \vec{\theta}) = f_\kappa(w)\theta^i - 2 \int_0^w dw' f_\kappa(w - w') \partial^i \Phi(f_\kappa(w)\beta^i(w'), w'). \quad (2.23)$$

We can express the previous using angular variables derivatives $\partial_\beta^i = f_\kappa(w)\partial^i$ to obtain

$$\beta^i(w, \vec{\theta}) = \theta^i - 2 \int_0^w dw' \frac{f_\kappa(w - w')}{f_\kappa(w)f_\kappa(w')} \partial_\beta^i \Phi(\beta^i(w'), w'). \quad (2.24)$$

Thin lens approximation. In typical strong gravitational lensing systems, there is a significant separation of scales: the distances between the observer, lens, and source are on the order of Gpc, while the physical size of the lens is on the order of kpc. This large separation of scales suggests the use of the so-called thin lens approximation. If we assume that we have just a thin lens at $w = w_1$, then $\Phi(\beta(w'), w') = \tilde{\Phi}(\beta(w_1))\delta(w' - w_1)$, where $\tilde{\Phi}$ is the projection of Φ over the thin lens plane. Then,

$$\beta^i(w, \vec{\theta}) = \theta^i - 2 \frac{f_\kappa(w - w_1)}{f_\kappa(w)f_\kappa(w_1)} \partial_\beta^i \tilde{\Phi}; \quad (2.25)$$

with $w = w_s$, we can express the previous using the lensing potential $\Psi(\vec{\theta})$, defined as

$$\Psi(\vec{\theta}) := 2 \frac{f_\kappa(w_s - w_1)}{f_\kappa(w_s)f_\kappa(w_1)} \int_{w_1^-}^{w_1^+} dw \Phi = 2 \frac{D_{ls}}{D_l D_s} \int \Phi(D_l \vec{\theta}, l) dl, \quad (2.26)$$

where on the last step we reconduce to the familiar form of the lensing potential with angular diameter distances; we exploited $D_A(z_j, z_i) = f_\kappa(w_j - w_i)/(1 + z_j)$ (Eq. (1.25)) and $dw = (1 + z_l) dl$ (l is meant to represent the proper distance near the lens plane and transverse to it). We defined $D_l := D_A(z_l, 0)$, $D_{ls} := D_A(z_s, z_l)$ and $D_s := D_A(z_s, 0)$.

Then, setting the displacement angle $\vec{\alpha}$ via

$$\nabla_{\vec{\theta}} \Psi(\vec{\theta}) = \vec{\alpha}(\vec{\theta}), \quad (2.27)$$

we recover the familiar lens equation

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}). \quad (2.28)$$

Exploiting the Poisson equation for Φ , we have (recall that the usual Poisson relation is found with proper distance derivatives)

$$\nabla_{\vec{\theta}}^2 \Psi(\vec{\theta}) = 2\kappa(\vec{\theta}), \quad \kappa := \frac{\Sigma(\vec{\theta})}{\Sigma_{cr}}, \quad \Sigma := \int \rho(D_l \vec{\theta}, l) dl, \quad \Sigma_{cr} := \frac{D_s}{4\pi G D_l D_{ls}}, \quad (2.29)$$

where $\kappa(\vec{\theta})$ is the convergence, $\Sigma(\vec{\theta})$ is the column density and Σ_{cr} is the critical column density.

2.2 Strong gravitational lensing basics

Typical angular displacement are of order 10^{-5} rad. With such small angles, one can linearize the lens equation Eq. (2.28) and describe distortions of images lying in the source plane due to the lens with

$$A_{ij} := \frac{\partial \beta_i}{\partial \theta_j} = \delta_{ij} - \partial_i \partial_j \Psi, \quad (2.30)$$

where we also exploited Eq. (2.29). The isotropic deformation is encoded in the trace,

$$\frac{1}{2} \text{tr} A = 1 - \frac{1}{2} \nabla^2 \Psi = 1 - \kappa ; \quad (2.31)$$

the non-isotropic deformations are encoded in the non-diagonal elements of A : calling γ the shear and φ the angle of preferred stretching, we can write

$$A = (1 - \kappa) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & -\cos 2\varphi \end{pmatrix} . \quad (2.32)$$

The determinant of A is connected to changes of volume (it is the Jacobian of the lens mapping Eq. (2.28)), so we can define the magnification due to lensing as

$$\mu := \frac{1}{\det A} = \frac{1}{(1 - \kappa)^2 - \gamma^2} . \quad (2.33)$$

From Eq. (2.29), we can write the lensing potential as a function of the convergence exploiting the 2D laplacian Green function

$$G(r) = \frac{1}{2\pi} \ln r \quad (2.34)$$

(r here denotes the 2D radial coordinate), as

$$\Psi(\vec{\theta}) = \frac{1}{\pi} \int d^2\theta' \ln |\vec{\theta} - \vec{\theta}'| \kappa(\vec{\theta}') , \quad (2.35)$$

and consequently, using Eq. (2.27), we can also write the deflection angle in terms of the convergence as

$$\vec{\alpha}(\vec{\theta}) = \frac{1}{\pi} \int d^2\theta' \frac{\vec{\theta} - \vec{\theta}'}{|\vec{\theta} - \vec{\theta}'|^2} \kappa(\vec{\theta}') . \quad (2.36)$$

Notice that, in case of spherical lenses ($\kappa(\vec{\theta}) = \kappa(\theta)$), the previous simplifies considerably; in fact, in that case Eq. (2.27) becomes

$$\frac{1}{\theta} \partial_\theta \psi + \partial_\theta^2 \psi = \frac{1}{\theta} \alpha + \partial_\theta \alpha = 2\kappa \implies \alpha = \frac{2}{\theta} \int_0^\theta \theta' \kappa(\theta') d\theta' . \quad (2.37)$$

The Schwarzschild lens model. Different lens models are characterized by assuming a functional form for the convergence profile, $\kappa(\vec{\theta})$. The simplest model is the point lens model, also called Schwarzschild model since it coincides with discussing gravitational lensing in the Schwarzschild metric (compare the following with the results in App. B, putting $H = 0$ there). We can characterize a point mass with mass M , sitting at the position $x = 0$, with the density profile

$$\rho(\vec{x}) = M \delta_D(\vec{x}) ; \quad (2.38)$$

the connection with the Schwarzschild lens comes from the relation $r_{\text{Sc}} = 2GM$, where r_{Sc} is the Schwarzschild radius. Then the convergence reads

$$\kappa(\vec{\theta}) = \frac{2\pi r_{\text{Sc}} D_1 D_{\text{ls}}}{D_s} \delta_D(D_1 \vec{\theta}) , \quad (2.39)$$

which, from Eq. (2.36) and Eq. (2.35), using the change of variable rule for the 1D Dirac delta $\delta_D(Ax) = \delta_D(x)/|A|$, implies the displacement angle and lensing potential

$$\vec{\alpha}(\vec{\theta}) = 2 \frac{r_{\text{Sc}}}{|\vec{\theta}|} \frac{D_{\text{ls}}}{D_1 D_{\text{ls}}} , \quad \Psi(\vec{\theta}) = 2 r_{\text{Sc}} \frac{D_{\text{ls}}}{D_1 D_{\text{ls}}} \ln |\vec{\theta}| . \quad (2.40)$$

The power-law model. Another simple but yet one of the most used models in gravitational lensing is the Power Law (PL) model. It assumes a density profile of the form

$$\rho(x) = \frac{\Sigma_{\text{cr}}}{D_1 \theta_E} \frac{3-\gamma}{2\sqrt{\pi}} \frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma-1}{2})} \left(\frac{x}{D_1 \theta_E} \right)^{-\gamma} , \quad (2.41)$$

where Γ is the Euler gamma function. This profile has two parameters: the PL slope γ and the normalization, fixed here by θ_E , the Einstein angle. The rather cumbersome prefactor is chosen such that formulas for lensing quantities greatly simplify, as we will see shortly. The surface density of this profile is

$$\Sigma = \frac{3-\gamma}{2} \Sigma_{\text{cr}} \left(\frac{\theta}{\theta_E} \right)^{1-\gamma} , \quad (2.42)$$

resulting with the displacement angle, convergence and lensing potential

$$\vec{\alpha}(\vec{\theta}) = \left(\frac{\theta}{\theta_E} \right)^{1-\gamma} \vec{\theta} , \quad \kappa(\vec{\theta}) = \frac{3-\gamma}{2} \left(\frac{\theta}{\theta_E} \right)^{1-\gamma} , \quad \Psi(\vec{\theta}) = \theta_E^{\gamma-1} \frac{\theta^{3-\gamma}}{3-\gamma} . \quad (2.43)$$

When considering an elliptic density profile, the expressions for lensing quantities become nasty. The usual approximation employed in the gravitational lensing literature is to introduce the ellipticity at the level of the convergence, rather than on the density. In practice, an elliptical PL profile is assumed to have the convergence [KSB94; Bar98]

$$\kappa(\vec{\theta}) = \frac{3-\gamma}{2} \left(\frac{\sqrt{q\theta_1^2 + \theta_2^2/q}}{\theta_E} \right)^{1-\gamma} , \quad (2.44)$$

where q is the axis ratio of the minor and major axes of the elliptical profile.

2.2.1 Time delays

The time a light ray takes to reach the observer from a source can be found using Fermat principle. In particular, as far as light propagation is concerned, we can think gravitational field to effectively behave as a medium with refraction index $n = 1/(1+2\Phi) \simeq 1-2\Phi$ (at first order in Φ); moreover, calling s the deflected path of the light ray, we can think about the actual path light takes on the geodesic ds as $ds = \cos(\hat{\alpha}) dl$, with dl the non-deflected path, and $\hat{\alpha}$ the instantaneous deflection angle. We then have, in the thin lens approximation

$$T = \int \frac{ds}{n} = \int \frac{n dl}{\cos \hat{\alpha}} \simeq \int dl (1-2\Phi) \left(1 + \frac{\hat{\alpha}^2}{2} \right) \simeq \int dl - \frac{D_1 D_s}{D_{\text{ls}}} \Psi + \frac{1}{2} \int \hat{\alpha}^2 dl ; \quad (2.45)$$

in the last equality, the first term is the time in unperturbed path, the second term represents the Shapiro-time delay and the third term comes from the geometric difference between the straight path l and the bent path s . Hence, we can write the light travel time as

$$T = T_{\text{unperturbed}} + T_{\text{Shapiro}} + T_{\text{geom}} . \quad (2.46)$$

Now, Fermat principle states that the actual path a light ray takes is the one which makes the travel time stationary with respect to infinitesimal changes in the path which keeps the extrema of the path itself fixed [SEF92]. In the small angle approximation (which for typical gravitational lensing systems always holds, since $\theta \sim 10^{-5}$), such path can be characterized by simply the observed angle $\vec{\theta}$, with fixed $\vec{\beta}$, and the actual path would correspond to the lens equation. Hence, Fermat principle states that

$$\nabla_{\theta} T(\vec{\beta}, \vec{\theta}) = 0 = -\frac{D_l D_s}{D_{ls}} \nabla_{\theta} \Psi(\vec{\theta}) + \nabla_{\theta} T_{\text{geom}}(\vec{\beta}, \vec{\theta}). \quad (2.47)$$

But $\nabla_{\theta} \Psi(\vec{\theta}) = \vec{\alpha}(\vec{\theta})$, hence, from the lens equation Eq. (4.13)

$$\nabla_{\theta} T_{\text{geom}}(\vec{\beta}, \vec{\theta}) = \frac{D_l D_s}{D_{ls}} (\vec{\theta} - \vec{\beta}) \implies T_{\text{geom}} = \frac{D_l D_s}{2D_{ls}} (\theta^2 - 2\vec{\theta} \cdot \vec{\beta}). \quad (2.48)$$

The time delay between two different images i and j ends up being (since the time delay arises at the lens, we have to consider also the gravitational redshift factor $1 + z_l$)

$$\Delta T_{ij} = D_{\Delta t} \left(\frac{\theta_i^2 - \theta_j^2 + 2\vec{\beta} \cdot (\vec{\theta}_j - \vec{\theta}_i)}{2} + \Psi(\vec{\theta}_j) - \Psi(\vec{\theta}_i) \right), \quad D_{\Delta t} := (1 + z_l) \frac{D_l D_s}{D_{ls}}. \quad (2.49)$$

The time delay distance $D_{\Delta t} \propto 1/H_0$ is sensitive to the Hubble constant, hence time delays, together with a lens model to extract the $\Delta t_{ij}/D_{\Delta t}$ factor, can be used to infer the Hubble constant [Ref64]. An explicit formula for this factor, assuming the validity of (1.22), is

$$D_{\Delta t} = \frac{1}{H_0 \Omega_{\Lambda,0}^{1/2}} \frac{r'(z_s, z_l) r'(z_s, 0)}{r'(z_l, 0)}, \quad (2.50)$$

where

$$r'(z_2, z_1) := \int_{z_1}^{z_2} \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}} (1 + z')^3 + 1 \right)^{-1/2} dz'. \quad (2.51)$$

2.2.2 Mass sheet degeneracy

The mass distribution $\kappa(\theta)$ and each of the distributions $\kappa_{\lambda}(\theta) := \lambda \kappa(\theta) + (1 - \lambda)$, together with an isotropic scaling of the source plane coordinates $\vec{\beta} \rightarrow \lambda \vec{\beta}$, yields exactly the same dimensionless observables, i.e., image positions, image shapes, magnification ratios, etc. This is the MSD. One can in fact easily verify that this transformation leaves the lens mapping Eq. (2.28) invariant. Some examples of quantities that are not invariant under the MSD are:

- The absolute magnifications. Using the fact that, for $\kappa = 1$ everywhere, we have $\Psi(\theta) = \theta^2/2 \implies \vec{\alpha} = \vec{\theta}$, we obtain (prime quantities refer to the MSD-transformed model)

$$A'_{ij} = \delta_{ij} - \lambda \frac{\partial \alpha'_i}{\partial \theta_j} = \delta_{ij} - \lambda \frac{\partial \alpha_i}{\partial \theta_j} - \delta_{ij} + \lambda \delta_{ij} = \lambda A_{ij} \implies \mu' = \mu / \lambda^2; \quad (2.52)$$

knowing the absolute magnification of a source can break the MSD.

- The time delays; they scale as $\Delta t'_{ij} = \lambda \Delta t_{ij}$, in fact,

$$\begin{aligned} \frac{\Delta T'_{ij}}{D_{\Delta t}} &= \lambda \vec{\beta} \cdot (\vec{\theta}_j - \vec{\theta}_i) + \lambda (\Psi_j - \Psi_i) + \frac{1}{2} (\theta_i^2 - \theta_j^2) + \frac{1}{2} (1 - \lambda) (\theta_j^2 - \theta_i^2) \\ &\implies \Delta T'_{ij} = \lambda \Delta T_{ij}; \end{aligned} \quad (2.53)$$

if one exploits Eq. (2.50) to measure H_0 , then one has

$$D_{\Delta t}^{\text{inferred}} = \Delta T_{ij}^{\text{measured}} / \left(\vec{\beta} \cdot (\vec{\theta}_j - \vec{\theta}_i) + \Psi_j - \Psi_i + \frac{1}{2}(\theta_i^2 - \theta_j^2) \right)^{\text{model}} \quad (2.54)$$

which implies

$$D_{\Delta t}^{\text{inferred}'} = D_{\Delta t}^{\text{inferred}} / \lambda \rightarrow H'_0 = \lambda H_0. \quad (2.55)$$

The MSD represents a possible serious systematic bias in the measurement of H_0 coming from time delays, as different degenerate models, given the same $\Delta t_{ij}^{\text{measured}}$, yield different H_0 measurements as can be seen from Eq. (2.55). We will extensively return back to this point in Ch. 4.

2.3 Multiple lens equation

We now generalize the lens equation Eq. (2.28) to the case of multiple sources, by taking into account LSS effects. This will be relevant for the discussion in Sec. 4.3.

Suppose we have multiple planes where strong gravitational lensing deflection takes place; in particular, suppose we have N main deflector planes located at different w_j , $j = 0, \dots, N-1$,

$$\Phi(\vec{\beta}(w), w) = \sum_{j=0}^{N-1} \tilde{\Phi}(\vec{\beta}(w_j), w_j) \delta(w - w_j) + \Phi_t(\vec{\beta}(w), w), \quad (2.56)$$

where $\Phi_t(\vec{\beta}(w), w)$ is the weak gravitational potential associated to LSS. We can approximate

$$\Phi_t(\vec{\beta}(w), w) \simeq \Phi_t(0, w) + \beta_i \partial_i^\beta \Phi_t(0, w). \quad (2.57)$$

Such an approximation goes with the name of tidal approximation, where the LOS part between different main deflector planes are seen as tidal planes. Within this approximation, we can write, from Eq. (2.24), the lens equation integral up to the first deflector sitting at w_0 as

$$\beta_i(w_0) = \theta_i - 2 \int_0^{w_0} dw' \frac{f_\kappa(w_0 - w')}{f_\kappa(w_0) f_\kappa(w')} \partial_i^\beta \Phi_t(0, w') - 2 \int_0^{w_0} dw' \frac{f_\kappa(w_0 - w')}{f_\kappa(w_0) f_\kappa(w')} \partial_i^\beta \partial_j^\beta \Phi_t(0, w') \beta_j(w'); \quad (2.58)$$

the second term on the RHS of the previous is an unobservable overall shift on the displacement angle (independent on $\vec{\theta}$), which can be reabsorbed in the source coordinates. Define

$$A_{ij}(w_2, w_1) := \delta_{ij} - M_{ij}(w_2, w_1) := \delta_{ij} - 2 \int_{w_1}^{w_2} dw' \frac{f_\kappa(w_2 - w') f_\kappa(w' - w_1)}{f_\kappa(w_2 - w_1) f_\kappa^2(w')} \partial_i^\beta \partial_j^\beta \Phi_t(0, w'). \quad (2.59)$$

We expect M_{ij} terms to be small since we are dealing with weak fields, hence we will consider only terms at first order in this quantity; in particular, substituting (for $w' < w_0$)

$$\beta_i(w') = \theta_i - 2 \int_0^{w'} dw'' \frac{f_\kappa(w' - w'')}{f_\kappa(w') f_\kappa(w'')} \partial_i^\beta \partial_j^\beta \Phi_t(0, w'') \beta_j(w''), \quad (2.60)$$

in Eq. (2.58), we see that at first order in M_{ij} , we should retain only the θ_i term of the previous, ending up with

$$\beta_i(w_0) = A_{ij}(w_0, 0) \theta_j. \quad (2.61)$$

For $w > w_0$, the situation changes due to the presence of the first strong deflector; from the full Φ of Eq. (2.57), where we don't use the tidal approximation for the strong deflector (just the thin lens

approximation), we have, up to w_1 ,

$$\begin{aligned}\beta_i(w_1) &= \theta_i - 2 \int_0^{w_1} dw' \frac{f_\kappa(w_1 - w')}{f_\kappa(w_1)f_\kappa(w')} \partial_i^\beta \partial_j^\beta \Phi_t(0, w') \beta_j(w') - 2 \frac{f_\kappa(w_1 - w_0)}{f_\kappa(w_1)f_\kappa(w_0)} \partial_i^\beta \tilde{\Phi}(\vec{\beta}(w_0)) \\ &= A_{ij}(w_1, 0) \theta_j - 2 \frac{f_\kappa(w_1 - w_0)}{f_\kappa(w_1)f_\kappa(w_0)} \partial_i^\beta \tilde{\Phi}(\vec{\beta}(w_0)) \\ &\quad + 2 \int_{w_0}^{w_1} dw' \frac{f_\kappa(w_1 - w')}{f_\kappa(w_1)f_\kappa(w')} \partial_i^\beta \partial_j^\beta \Phi_t(0, w') \left[2 \frac{f_\kappa(w' - w_0)}{f_\kappa(w_0)f_\kappa(w')} \partial_i^\beta \tilde{\Phi}(\vec{\beta}(w_0)) \right],\end{aligned}\tag{2.62}$$

where on the last step we substituted $\beta_j(w')$ inside the integral with the term

$$\beta_i(w') = \begin{cases} \theta_i & \text{for } w \leq w_0, \\ \theta_i - 2 \frac{f_\kappa(w' - w_0)}{f_\kappa(w_0)f_\kappa(w')} \partial_i^\beta \tilde{\Phi}(\vec{\beta}(w_0)) & \text{for } w > w_0, \end{cases}\tag{2.63}$$

so that we always remain at first order in M_{ij} . For later convenience, we define

$$\vec{\alpha}_j(\vec{\beta}(w_j)) := \frac{f_\kappa(w_N - w_j)}{f_\kappa(w_N)f_\kappa(w_j)} \partial_j^\beta \tilde{\Phi}(\vec{\beta}(w_j)), \quad C_{ij} := \frac{f_\kappa(w_i - w_j)f_\kappa(w_N)}{f_\kappa(w_N - w_j)f_\kappa(w_i)};\tag{2.64}$$

with these definitions, the displacement angles $\vec{\alpha}_j(\vec{\beta}(w_j))$ are normalized with respect to the final source sitting at w_N .

Notice that we can rewrite the term in square brackets in Eq. (2.62) as

$$2 \frac{f_\kappa(w' - w_0)}{f_\kappa(w_0)f_\kappa(w')} \partial_i^\beta \tilde{\Phi}(\vec{\beta}(w_0)) = \frac{f_\kappa(w' - w_0)f_\kappa(w_1)}{f_\kappa(w')f_\kappa(w_1 - w_0)} C_{10} \alpha_{0i}(\vec{\beta}(w_0)),\tag{2.65}$$

and we finally end up with [BK96]

$$\beta_i(w_1) = A_{ij}(w_1, 0) \theta_j - A_{ij}(w_1, w_0) C_{10} \alpha_{0j}(A_{ij}(w_0, 0) \theta_j).\tag{2.66}$$

The reasoning for the inclusion of the deflection in the other main deflector planes is very similar, leading to the multiple sources lens equation

$$\vec{\beta}(w_j) = A(w_j, 0) \vec{\theta} - \sum_{i=0}^{j-1} A(w_j, w_i) C_{ji} \vec{\alpha}_i(\vec{\beta}(w_i)).\tag{2.67}$$

Time delays with LSS and multiple sources. The time delay between image solutions of Eq. (2.66) can be computed by exploiting the Fermat principle [SEF92; Sch97; Sch19], similar to what we did in Sec. 2.2.1. First, we can try to recover the term of the time delay which depends on the lensing potential only. Note that, from the metric Eq. (2.9), we can write, for a main deflector at redshift z_i

$$\frac{dt}{dl} = 1 - 2\Phi \implies \Delta T_i^{\text{Shapiro}} = -2 \int_{\Delta l_i} \Phi dl,\tag{2.68}$$

where l is the proper distance at redshift z_i and Δl_i is the extension of the deflector. When summing over all the lenses, the time delay as seen by the observer at $w = 0$ reads

$$\Delta T^{\text{Shapiro}} = -2 \sum_i (1 + z_i) \int_{\Delta l_i} \Phi dl,\tag{2.69}$$

where the factor $(1 + z_i)$ takes into account cosmological redshift effects. With the definition Eq. (2.64) and the corresponding $\nabla_{\vec{\beta}} \Psi_i = \vec{\alpha}_i(\vec{\beta})$, we can write

$$\Psi_i = 2 \frac{f_\kappa(w_N - w_i)}{f_\kappa(w_N) f_\kappa(w_i)} \int_{\Delta w_i} \Phi \, dw = 2 \frac{D_{Ni}}{D_N D_i} \int_{\Delta l_i} \Phi \, dl, \quad (2.70)$$

where we exploited Eq. (1.25) for the angular diameter distances and $dw = (1 + z_i) dl$. We thus can write Eq. (2.69) as

$$\Delta T^{\text{Shapiro}} = - \sum_i D_{\text{dt}}^i \Psi_i(\vec{\beta}_i), \quad D_{\text{dt}}^i := (1 + z_i) \frac{D_N D_i}{D_{Ni}}. \quad (2.71)$$

The Fermat principle states that, up to an affine transformation, the lens equation for a single source can be obtained by taking the gradient $\nabla_{\vec{\theta}}$ of the light travel time function $T(\vec{\theta}, \vec{\beta})$ and setting it to zero. Eq. (2.71) can then be used to understand what is the correct prefactor (the affine parameter) entering the light travel time function. For a single lens system, we see that from the function

$$\begin{aligned} \Delta T(\vec{\theta}, \vec{\beta}) &= D_{\text{dt}} \left(\frac{1}{2} \vec{\theta}^T (\mathbb{I} - M(w_s, 0) - M(w_l, 0) + M(w_s, w_l)) \vec{\theta} \right. \\ &\quad \left. - \vec{\beta}^T (\mathbb{I} - M(w_l, 0) + M(w_s, w_l)) \vec{\theta} - \psi((\mathbb{I} - M(w_l, 0)) \vec{\theta}) \right), \end{aligned} \quad (2.72)$$

one indeed recovers Eq. (2.66) using $\nabla_{\vec{\theta}} \Delta T(\vec{\theta}, \vec{\beta}) = 0$, recalling the definition $\nabla_{\vec{\theta}} \Psi(\vec{\theta}) = \vec{\alpha}(\vec{\theta})$. The generalization to multiple lens planes, found recursively via $\nabla_{\vec{\beta}_i} \Delta T(\{\vec{\beta}_i\}, \vec{\beta}_N) = 0$ to yield the $i + 1$ -th equation of the multiple lens equation Eq. (2.67), reads [Sch19]

$$\Delta T(\vec{\theta}, \{\vec{\beta}_i\}) = \sum_{i=0}^{N-1} \frac{D_{\text{dt}}^i}{2 C_{i+1,i}} \left(\vec{B}_i^T A_{i+1,i}^{-1} A_{i+1} A_i^{-1} \vec{B}_i - \Psi_i(\vec{\beta}_i) \right), \quad \vec{B}_i := (A_{i+1,i}^{-1} A_{i+1} A_i^{-1} \vec{\beta}_i - A_{i+1,i}^{-1} \vec{\beta}_{i+1}), \quad (2.73)$$

where we used the shorthands $A_{ij} := A(w_i, w_j)$, $A_i := A(w_i, 0)$.

2.4 External convergence and shear

The matrices $M(w_2, w_1)$ introduced in the previous section encode external convergence and shear coming from LSS between comoving distance w_1 and w_2 . To make this apparent, we can write them in the following way

$$\mathbb{I} - M(w_2, w_1) =: \begin{pmatrix} 1 - \kappa(w_2, w_1) + \gamma_1(w_2, w_1) & \gamma_2(w_2, w_1) \\ \gamma_2(w_2, w_1) & 1 - \kappa(w_2, w_1) - \gamma_1(w_2, w_1) \end{pmatrix}, \quad (2.74)$$

where $\kappa(w_2, w_1)$, $\gamma_i(w_2, w_1)$ are the external convergence and shears between metric comoving distances w_1 and w_2 . Explicitly,

$$\kappa = \frac{1}{2} (M_{11} + M_{22}), \quad \gamma_2 = M_{12}, \quad \gamma_1 = \frac{1}{2} (M_{11} - M_{22}). \quad (2.75)$$

We want now to find an expression for $\kappa(w_2, w_1)$ given a cosmology.

To simplify the notation, we define

$$q_{kl}(w) := \Theta(w - w_l) \Theta(w_k - w) \frac{f_\kappa(w_k - w) f_\kappa(w - w_l)}{f_\kappa(w_k - w_l)} (1 + z(w)); \quad (2.76)$$

recall

$$\frac{\nabla_{\vec{\theta}}^2 \Phi}{w^2} = \nabla_{\vec{x}}^2 \Phi = 4\pi G a^2 \rho_m \delta = \left(\frac{3H^2}{2} - 4\pi G \rho_\Lambda \right) \delta = \frac{3H_0^2 \Omega_{m,0}}{2} a^{-3}, \quad (2.77)$$

from which we have

$$k(w_2, w_1; \vec{\theta}) = \frac{3H_0^2 \Omega_{m,0}}{2} \int dw q_{21}(w) \delta((\vec{\theta}, w), w), \quad (2.78)$$

where w labels both the spatial and the time coordinate, and z is the redshift. From this expression, it is clear that the mean value $\langle \kappa \rangle = 0$ due to homogeneity and isotropy, so we are driven to deal with

$$\begin{aligned} \langle k(w_i, w_j; \vec{\theta}) k(w_l, w_m; \vec{\theta}') \rangle &= \frac{9H_0^4 \Omega_{m,0}^2}{4} \int dw \int dw' q_{ij}(w) q_{lm}(w') \\ &\times \langle \delta((\vec{\theta}, w), w) \delta((\vec{\theta}', w'), w') \rangle; \end{aligned} \quad (2.79)$$

we now pass to Fourier space and use the power spectrum expression

$$\langle \delta(\vec{k}, w) \delta(\vec{k}', w') \rangle = (2\pi)^3 \delta_D(\vec{k} + \vec{k}') P_\delta(k, w, w'), \quad (2.80)$$

(notice that $\vec{k}$ is a 3D vector, so here w is meant to denote just the time dependence) to arrive at

$$\begin{aligned} \langle k(w_i, w_j; \vec{\theta}) k(w_l, w_m; \vec{\theta}') \rangle &= \frac{9H_0^4 \Omega_{m,0}^2}{4} \int dw \int dw' q_{ij}(w) q_{lm}(w') \\ &\times \int \frac{d^3 k}{(2\pi)^3} P_\delta(k, w, w') e^{-i\vec{k}_\perp \cdot (w\vec{\theta} - w'\vec{\theta}')} e^{-ik_3(w - w')}, \end{aligned} \quad (2.81)$$

where k_3 denotes the component of $\vec{k}$ along the line of sight and $\vec{k}_\perp$ the transverse one. To simplify this expression, we can use the expansion

$$e^{i\vec{k} \cdot \vec{x}} = (4\pi) \sum_{l,m} i^l Y_{lm}^*(\hat{k}) Y_{lm}(\hat{x}) j_l(kx), \quad (2.82)$$

where Y_{lm} are spherical harmonics and j_l are spherical Bessel functions. We can write (exploiting $Y_{lm}(-\hat{x}) = (-1)^l Y_{lm}(\hat{x})$)

$$\begin{aligned} e^{-i\vec{k}_\perp \cdot (w\vec{\theta} - w'\vec{\theta}')} e^{-ik_3(w - w')} &= e^{i\vec{k} \cdot \hat{n}' w'} e^{-i\vec{k} \cdot \hat{n} w} \\ &= (4\pi)^2 \sum_{lm l' m'} i^l i^{l'} (-1)^{l'} Y_{lm}^*(\hat{k}) Y_{lm}(\hat{n}) j_l(kw) Y_{l' m'}^*(\hat{n}') Y_{lm}(\hat{k}) j_{l'}(kw'). \end{aligned} \quad (2.83)$$

Upon the solid angle integration on the variable $\vec{k}$, we can exploit the spherical harmonics orthogonality relation to end up with

$$\int d\Omega_k e^{i\vec{k} \cdot \hat{n}' w'} e^{-i\vec{k} \cdot \hat{n} w} = (4\pi)^2 \sum_{lm} Y_{lm}^*(\hat{n}') Y_{lm}(\hat{n}) j_l(kw) j_l(kw'). \quad (2.84)$$

The correlator will then read

$$\begin{aligned} \langle k(w_i, w_j; \vec{\theta}) k(w_l, w_m; \vec{\theta}') \rangle &= \frac{9H_0^4 \Omega_{m,0}^2}{2\pi} \int dw \int dw' q_{ij}(w) q_{lm}(w') \\ &\times \int dk k^2 P_\delta(k, w, w') \sum_{lm} Y_{lm}^*(\hat{n}') Y_{lm}(\hat{n}) j_l(kw) j_l(kw'). \end{aligned} \quad (2.85)$$

In the case where the typical length scale change in w of the power spectrum is much larger than the oscillations in Bessel functions, then one can exploit the relation

$$\int_0^\infty dx x^2 j_l(ux) j_l(vx) = \frac{\pi}{2u^2} \delta_D(u - v) , \quad (2.86)$$

to end up with (evaluating P_δ at l/w , where the l -th Bessel function peaks)

$$\langle k(w_i, w_j; \vec{\theta}') k(w_l, w_m; \vec{\theta}) \rangle = \frac{9H_0^4 \Omega_{m,0}^2}{4} \int dw \frac{q_{ij}(w) q_{lm}(w)}{w^2} \sum_{lm} Y_{lm}^*(\hat{n}') Y_{lm}(\hat{n}) P_\delta\left(\frac{l}{w}, w\right) . \quad (2.87)$$

This corresponds to the Limber's approximation [Lim53]. A different expression comes from assuming the possibility to set $\hat{n} = \hat{n}'$ (same LOS approximation). To marginally test how good this approximation could be, we can split

$$\exp\left(i\vec{k} \cdot (\hat{n}' w' - \hat{n} w)\right) = \exp\left(i\vec{k} \cdot \hat{n}(w' - w)\right) \exp\left(iw' \vec{k} \cdot (\hat{n}' - \hat{n})\right) , \quad (2.88)$$

where the correction is given by the last exponential term of the previous. Considering a small angular displacement, it is easy to see that we can write $\hat{n}' - \hat{n} = \delta\hat{n}\delta\theta$, with $\delta\hat{n}$ a versor perpendicular to $\hat{n}$. Then, the entity of the correction will be given by $\exp(i\delta\hat{n}\delta\theta w k)$; in particular, the largest correction will be given by the part of the integral at large w and k . For a typical weak lensing setting, we can set $w \sim 10^3$ Mpc and $k \sim 10^2$ Mpc $^{-1}$ as the possible large correction region values (for larger k , the integrand kills the correction anyway). With typical source displacements of $\delta\theta \sim 10^{-6}$ rad, we have that the exponent is of the order of 10^{-1} , making the exponential always very near unity. Notice that with this assumption, the result we obtain is identical to the one that is obtained with the flat sky approximation [BS01].

Assuming then $\hat{n} = \hat{n}'$, we can use

$$\sum_m Y_{lm}^*(\hat{n}) Y_{lm}(\hat{n}) = \frac{2l+1}{4\pi} , \quad j_0(|\vec{p} - \vec{p}'|r) = \sum_{l=0}^{\infty} (2l+1) j_l(pr) j_l(p'r) P_l(\cos\theta) , \quad (2.89)$$

where here $\cos\theta = 1$ (θ is the angle between $\vec{p}$ and $\vec{p}'$), and $P_l(1) = 1$ for all l . We end up with the following external convergence expectation value

$$\begin{aligned} \langle \kappa^2(w_2, w_1) \rangle &= \frac{9H_0^4 \Omega_{m,0}^2}{2} \frac{1}{(2\pi)^2} \int dw \int dw' q_{21}(w) q_{21}(w') \\ &\quad \times \int dk k^2 j_0(k(w - w')) P_\delta(k, w, w') . \end{aligned} \quad (2.90)$$

The non-equal time power spectrum $P_\delta(k, w, w')$ can then be computed using Eq. (1.75). The discussion of this section will be relevant for the work Ref. [Teo+22], which will be the topic of Sec. 4.3.

Chapter 3

Ultralight Dark Matter

ULDM is a well-motivated DM candidate, potentially arising in high energy completions of the standard model of particle physics. It is generically produced in the early universe via the vacuum misalignment mechanism, and is stable on cosmological timescales [PWW83; AS83; DF83; SW06; Arv+10; Mar16] and the scenario of particle mass as low as $m < 10^{-20}$ eV has been discussed extensively in the literature [HBG00; Hui+17; Hui21; Fer21]. A plethora of axion-like particles spanning a range of masses may occur in string theory [SW06; Arv+10; Mar16], motivating the possibility of several ULDM fields, each contributing a fraction of the DM budget.

Compared to CDM, ULDM leads to novel behaviour on distances comparable to or smaller than its de-Broglie wavelength

$$\lambda = \frac{2\pi}{mv} \sim 6 \text{ pc} \left(\frac{1 \times 10^{-20} \text{ eV}}{m} \right) \left(\frac{2 \times 10^2 \text{ km s}^{-1}}{v} \right), \quad (3.1)$$

where v is the characteristic velocity of a system. On such scales, the ULDM wave-like nature is manifest. This distinct wave behaviour makes possible to constrain the ULDM mass-fraction parameter space using gravitational probes alone, without relying on model-dependent assumptions (e.g. self-interactions or couplings to the standard model).

This wave behaviour results in a suppression of power in cosmological perturbations, leaving observable imprints on the CMB anisotropy power spectrum, galaxy clustering [Lag+22], and the Lyman-alpha forest [Ir17; Arm+17; Kob+17; Leo+19]. ULDM wave-like density fluctuations can also lead to astrophysical effects inside galaxies, such as dynamical heating and dynamical friction [Hui+17; Lan+20; DC+23], leading to constraints using systems like dwarf and ultra-faint dwarf galaxies [Zim+24; TCB25; DK22].

Another prominent feature of ULDM in galaxies is the formation of cored profiles in the inner part of the dark matter halo [SCB14; Sch+14; Hui+17]. These cores, called “solitons” in the ULDM literature, are the ground state solution of the Schrödinger-Poisson equations, obeyed by ULDM in the non-relativistic regime. They are consistently seen in simulations, and some of their key features are understood theoretically [Cha11; CD11; LPT18; GHPW15; BOFT19; BOFT21; CSX24; CSX23].

Gravitational lensing offers another probe of ULDM. Magnification anomalies can be used to constrain the amount of substructure present in a galaxy [Koo05; Xu+13; VK09; Veg+23], including substructure due to ULDM [Lar+22; Pow+23]. Gravitational lensing is also sensitive to ULDM cores via time delays, as we will explain in Ch. 5.

In the following, we describe basic features of the ULDM paradigm, to be used in Refs. [BT21b; BT25; TCB25; Blu+25], works this thesis is based on. We start deriving the equations ULDM obeys to, i.e. the Schrödinger-Poisson Equations (SPE), in Sec. 3.1; we describe the main features of ULDM we briefly introduced above in Sec. 3.2 and discuss the various constraints of the ULDM paradigm in Sec. 3.3.

3.1 The Schrödinger-Poisson equations for ULDM

In the most minimal model, ULDM is a scalar field ϕ minimally coupled to gravity. Its action reads

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 \right); \quad (3.2)$$

the metric we assume is the first order perturbed flat Robertson-Walker metric in Newtonian gauge (see also App. C.3.1), neglecting tensor and residual vector modes,

$$ds^2 = -(1 + 2\Phi) dt^2 + a^2(t)(1 - 2\Phi) d\vec{x}^2. \quad (3.3)$$

By varying the action with respect to the field, one can obtain the equation of motion

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = m^2 \phi. \quad (3.4)$$

To first order in Φ and neglecting its time variation (weak field limit), recalling $\sqrt{-g} = (a^6(1 + 2\Phi)(1 - 2\Phi)^3)^{1/2} \simeq a^3(1 - 2\Phi)$, we have

$$-3H(1 - 2\Phi) \partial_0 \phi - (1 - 2\Phi) \partial_0^2 \phi + (1 + 2\Phi) \frac{\nabla^2 \phi}{a} = m^2 \phi, \quad (3.5)$$

where $H = \dot{a}/a$.

ULDM is a Bose system with very high occupation number in phase space; we can thus approximate it using a single wave function describing a classical field, since all interactions besides gravity can be neglected. Quantum corrections $\delta\hat{\phi}$ are expected to be completely negligible, since

$$\delta\hat{\phi} \sim \frac{1}{\sqrt{\mathcal{N}}}, \quad \mathcal{N} \sim \frac{\rho_m}{m(mv)^3} \sim 10^{84} \left(\frac{\rho_m}{0.4 \text{ GeV cm}^{-3}} \right) \left(\frac{1 \times 10^{-20} \text{ eV}}{m} \right)^4, \quad (3.6)$$

where 0.4 GeV cm^{-3} is the inferred DM density in the Milky Way on radial scales of around 10 kpc. In other words, we are assuming the validity of mean field theory, where the mean values of operators are large compared to the root variance around those values due to quantum corrections, this root variance going as $\sim 1/\sqrt{\mathcal{N}}$.

We want to describe ULDM in galactic systems, where typical particle velocities $v \ll 1$, meaning we are in the Non Relativistic (NR) regime; due to this reason, it will be useful to write the field with respect to the NR field ψ

$$\phi = \frac{1}{\sqrt{2m}} (\psi e^{-imt} + \psi^* e^{imt}), \quad (3.7)$$

where we factored out the fast variations coming from the mass m , exploiting the dispersion relation

$$E \sim \sqrt{m^2 + p^2} \simeq m \left(1 + \frac{v^2}{2} \right); \quad (3.8)$$

ψ is a complex scalar field, with the dimensions of a mass square (since the real scalar field ϕ must have dimensions of a mass). Plugging the previous in (3.5), neglecting terms exploiting the NR limit condition $|\partial_0 \psi|/(m|\psi|) \sim v^2 \ll 1$, we obtain the Schrödinger-like equation

$$i\partial_0 \psi + \frac{3}{2} iH\psi = -\frac{\nabla^2 \psi}{2ma^2} + m\Phi\psi. \quad (3.9)$$

In the following, since we are primarily interested in galactic dynamics (length scale ~ 100 kpc), we can safely neglect the Hubble term and universe expansion effects, so we will set $a = 1$ and $H = 0$.

To obtain the equation for the gravitational potential, we can use the 00 Einstein equation in the weak field limit, which reads (from Eq. (C.41) with $a(t) = 1$):

$$\nabla^2 \Phi = 4\pi G \delta T_{00} ; \quad (3.10)$$

notice that Φ is sourced only by the fluctuations over the ULDM cosmological background; the cosmological background will just enter Friedmann equations, as can be seen in the derivation of Sec. C.3.1.

The energy momentum tensor reads

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi + \frac{1}{2} m^2 \phi^2 \right) , \quad (3.11)$$

which yields $\delta T_{00} = |\psi|^2 - \langle |\psi|^2 \rangle$ in the NR limit.

In the end, we end up with the Schrödinger-Poisson Equations (SPE)

$$i\partial_0 \psi = -\frac{\nabla^2 \psi}{2m} + m\Phi \psi , \quad (3.12)$$

$$\nabla^2 \Phi = 4\pi G (|\psi|^2 - \langle |\psi|^2 \rangle) . \quad (3.13)$$

The SPE can be generalized to the n fields ψ_i case, each with mass m_i , plus a possible source of external gravitational potential Φ_{ext} (possibly coming from stars or other DM components), not sourced by ψ_i , for which back-reaction is neglected. Such a generalization brings us to

$$i\frac{\partial \psi_i}{\partial t} = -\frac{1}{2m_i} \nabla^2 \psi_i + m_i (\Phi + \Phi_{\text{ext}}) \psi_i , \quad (3.14)$$

$$\nabla^2 \Phi = 4\pi G \sum_i (|\psi_i|^2 - \langle |\psi_i|^2 \rangle) . \quad (3.15)$$

3.2 Features of ULDM

Cores. The ground state solution of the SPE, called soliton, is a cored solution. Soliton formation within ULDM haloes has been identified in numerical simulations by different groups [GUL04; SCB14; Sch+14; SNE16; VN16; Moc+17; VNS18; LPT18; EN19; Che+21; Sch+20], and is consistent with analytic considerations which show that the soliton is an attractor solution of the equations of motion [Cha11]. We now study the features of this solution.

To find the soliton solution, write ψ as

$$\psi(\vec{x}, t) = \frac{m}{\sqrt{4\pi G}} e^{-i\gamma m t} \chi(\vec{x}) , \quad (3.16)$$

where γ is an eigenvalue of the problem. Assume spherical symmetry and define the dimensionless radius $\tilde{x} := xm$, then the SPE become

$$\partial_{\tilde{x}}^2 \chi + \frac{2}{\tilde{x}} \partial_{\tilde{x}} \chi = 2(\Phi + \Phi_{\text{ext}} - \gamma) \chi , \quad (3.17)$$

$$\partial_{\tilde{x}}^2 \Phi + \frac{2}{\tilde{x}} \partial_{\tilde{x}} \Phi = \chi^2 . \quad (3.18)$$

We include the static background gravitational potential Φ_{ext} , coming e.g. from stars and from other (non-ULDM) contributions to the DM. Such a background potential will be relevant for the setting described in Ch. 5. The solution is fixed by a single parameter that we can take to be the value of χ at $r = 0$. We thus define the solution $\chi_\lambda(r)$ via

$$\chi_\lambda(r=0) = \lambda_s^2, \quad (3.19)$$

with a real parameter λ_s . It is convenient to use the scaling relation [Bar+18]

$$\chi_\lambda(r; \Phi_{\text{ext}}(\cdot)) = \lambda_s^2 \chi_1(\lambda_s r; \lambda_s^{-2} \Phi_{\text{ext}}(\lambda_s^{-1} \cdot)), \quad (3.20)$$

meaning that in numerical investigations, it is always enough to compute χ_1 . For clarity, in Eq. (3.20) we explicitly note how the external potential enters the solution.

The ground energy state solution correspond to the one with no nodes, which happens to be a cored solution with characteristic core radius of order astrophysical scales. It is possible to easily solve these equations numerically, by assuming $\chi(0) = \lambda_s^2$ as initial condition, together, for consistence with the form of the equations, with $\partial_{\tilde{x}}\chi = \partial_{\tilde{x}}\Phi = 0$; the initial condition for $\Phi(0)$ can be found by selecting the value that allows for a no-node solution with $\chi(\tilde{x} \rightarrow \infty) \rightarrow 0$, whereas γ is fixed by imposing $\Phi(\tilde{x} \rightarrow \infty) \rightarrow 0$. Numerical calculations for the $\Phi_{\text{ext}} = 0$ soliton, which we call the self-gravitating soliton, show that, for $\lambda_s^2 = 1$, $\gamma_1 \simeq -0.69$, where the subscript 1 indicates the solution corresponding to the initial condition $\lambda_s^2 = 1$. Further details of the soliton solution can be found in App. A.2. The mass of the χ_1 self-gravitating soliton is

$$M_1 = \int d^3x |\psi^2| = \frac{M_{\text{pl}}^2}{m} \int_0^\infty d\tilde{x} \tilde{x}^2 \chi_1^2(\tilde{x}) \simeq 2.79 \times 10^{11} \left(\frac{m}{10^{-21} \text{eV}} \right)^{-1} M_\odot. \quad (3.21)$$

A good measure of the core size is the core radius, defined as the one where the density drops by half. It reads

$$r_{c1} \simeq 8.2 \times 10^{-4} \left(\frac{m}{10^{-21} \text{eV}} \right)^{-1} \text{pc}. \quad (3.22)$$

We have $M_\lambda = \lambda_s M_1$, $r_{c\lambda} = \lambda_s^{-1} r_{c1}$, so their product is independent of λ_s , and yields

$$M_\lambda r_{c\lambda} \simeq 2.27 \times 10^{10} \left(\frac{m}{10^{-21} \text{eV}} \right)^{-2} \text{kpc} M_\odot, \quad (3.23)$$

which allows us to rewrite

$$r_{c\lambda} \simeq 0.3 \left(\frac{m}{10^{-21} \text{eV}} \right)^{-2} \left(\frac{M}{10^9 M_\odot} \right)^{-1} \text{kpc}. \quad (3.24)$$

$r_{c\lambda}$ is a good measure of the core size, which indeed reaches astrophysical scales as anticipated.

It can be useful to write down an analytic approximation for the self-gravitating soliton density, first used in Ref. [Sch+14],

$$\rho_\lambda(r) = \frac{1.9 \left(\frac{10^{-21} \text{eV}}{m} \right)^2 \left(\frac{100 \text{pc}}{r_{c\lambda}} \right)^4}{\left(1 + 0.091 \left(\frac{r}{r_{c\lambda}} \right)^2 \right)^8} M_\odot \text{pc}^{-3}. \quad (3.25)$$

The energy can be computed as, with $x' := m\lambda_s r$

$$E_{\lambda_s} = \frac{m^2 M_{\text{PL}}^2}{4\pi} \lambda_s^6 \int \frac{d^3 x'}{m^3 \lambda_s^3} \left(\frac{1}{2} |\nabla' \chi_1(x')|^2 + \frac{\Phi_1(x') |\chi_1|^2(x')}{2} \right) \quad (3.26)$$

$$\implies E_{\lambda_s} = \lambda_s^3 E_1 = \frac{\lambda_s^3}{3} \gamma_1 M_1 ;$$

the last equality is obtained using the virial theorem, $E_{\text{pot}} = -2E_{\text{kin}}$, valid for the stationary SPE [Hui+17].

For a non-self gravitating soliton, with $\Phi_{\text{ext}} \neq 0$, Eq. (3.25) must be modified, depending on the form of Φ_{ext} . We will see an example in Ch. 5.

Fluctuating density field. The validity of the wave description implies that the density field itself experiences wave phenomena like interference. In general, a wavefunctions can be expressed as a superposition of eigenmodes ψ_j with amplitudes a_j (we can take a_j to be real)

$$\psi(\vec{x}, t) = \sum_j a_j \psi_j(\vec{x}) e^{i\varphi_j} e^{-i\omega_j t} , \quad (3.27)$$

where ω_j is the energy associated to the eigenmode ψ_j and φ_j is a random phase. This field configuration corresponds to the density

$$\rho(\vec{x}) = |\psi|^2 = \sum_j |a_j|^2 |\psi_j(\vec{x})|^2 + \sum_{j \neq k} a_j a_k \psi_j \psi_k^* e^{i(\varphi_j - \varphi_k)} e^{-it(\omega_j - \omega_k)} , \quad (3.28)$$

where the first term in the rightmost hand side is the average density, and the last term is the interference term.

The ULDM fluctuating density field has important phenomenological consequences. In particular, the kinetic properties of the fluctuating ULDM field can be seen as equivalent to a classical N body system composed of quasi-particles with effective mass M_{QP} [Hui+17; BOFT19]

$$M_{\text{QP}} = \rho \lambda_{\text{dB}}^3 \sim 2 \times 10^6 M_{\odot} \left(\frac{\rho}{0.01 M_{\odot} / \text{pc}^3} \right) \left(\frac{10^{-21} \text{ eV}}{m} \right)^3 \left(\frac{10 \text{ km s}^{-1}}{\sigma} \right)^3 , \quad (3.29)$$

where σ is the velocity dispersion. From this analogy, one can use the expression for quantities obtained in kinetic theory for N body systems and convert then to expressions valid for ULDM by substituting $m \rightarrow M_{\text{QP}}$; for example, the relaxation time of a ULDM halo reads [BOFT19]

$$t_{\text{rel}} \sim \frac{m^3 \sigma^6}{G^2 \rho^2 \ln \Lambda_C} , \quad (3.30)$$

where Λ_C is the Coulomb logarithm. For typical m and σ values, M_{QP} is much larger than the typical mass of stars within a galaxy. A consequence of this fact is that quasi-particles inject energy on a stellar system. Such a process is called dynamical heating, and it will be the primary way we use to give a constraint on the allowed ULDM particle mass in Sec. 6.1. The fluctuating density field can be seen to be also the responsible of soliton quasinormal mode oscillations [GUL04; CSX23] and soliton random walk [LHY21], which also contribute to stars dynamical heating.

“Quantum pressure” and small scales suppression. We can recast the SPE as continuity and Euler equation from fluid dynamics, using the so-called Madelung formulation [Mad27]

$$\psi = \sqrt{\frac{\rho}{m}} e^{i\theta}, \quad \vec{v} = \frac{1}{m} \nabla \theta = \frac{1}{2m|\psi|^2} (\psi_i^* \nabla \psi_i - \psi_i \nabla \psi_i^*), \quad (3.31)$$

to obtain

$$\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0, \quad (3.32)$$

$$\partial_t \vec{v} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla \Phi + \frac{1}{2m^2} \nabla \left(\frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \right). \quad (3.33)$$

We remark that such a formulation does not describe vortices [Hui+21], which are configurations where the density vanishes in one point (where the Madelung formulation Eq. (3.31) loses its meaning).

Notice that Eqs.(3.32), (3.33) are completely classical, and that we can loosely identify, in analogy with how the Euler equation Eq. (1.48) looks like,

$$\frac{1}{2m^2} \frac{\nabla^2 \sqrt{\rho}}{\sqrt{\rho}} \sim P_{\text{quantum}}, \quad (3.34)$$

which is usually called quantum pressure. This is unfortunately a misnomer on multiple levels. One, as we remarked, the equations are classical, so there is no real quantum behaviour. Second, it is not really a pressure, since it does not come from the isotropic part of the T_{ij} tensor. Nevertheless, this name gives intuition on its effects: it acts like an outward pressure, which slows down gravitational collapse, and whose origin can be traced back to the uncertainty principle (one cannot compress too much matter in a small volume: the more it is done, the more the position of particles becomes precise, and hence the more the velocity dispersion has to increase).

To understand at least qualitatively this effect of resistance to gravitational collapse (which will end up in small scales power spectrum suppression), we can consider to expand Eqs. (3.32), (3.33) in $\rho = \bar{\rho}_m(1 + \delta)$, with $|\delta| \ll 1$, and $|\vec{v}| \ll 1$. At first order in δ and $\vec{v}$, we have

$$\dot{\vec{v}} = -\nabla \Phi + \frac{1}{2m^2} \nabla \frac{\nabla^2 \delta}{2}, \quad \dot{\delta} + \nabla \cdot \vec{v} = 0; \quad (3.35)$$

taking the derivative of the latter and using the former, together with the Poisson equation¹

$$\nabla^2 \Phi = 4\pi G \bar{\rho} \delta, \quad (3.36)$$

we have

$$\ddot{\delta} - 4\pi G \bar{\rho} \delta + \frac{1}{4m^2} \nabla^4 \delta = 0. \quad (3.37)$$

The previous is solved by

$$\delta = A_1 e^{i(\omega t - \vec{k} \cdot \vec{x})} + A_2 e^{-i(\omega t - \vec{k} \cdot \vec{x})}, \quad (3.38)$$

with

$$\omega^2 = \frac{1}{4m^2} k^4 - 4\pi G \bar{\rho}; \quad (3.39)$$

this defines the so called Jeans scale

$$k_J = (16\pi G \bar{\rho})^{1/4} m^{1/2}. \quad (3.40)$$

¹The Newtonian potential Φ is sourced only by the fluctuations, see Sec. 1.1.4. The mean density $\bar{\rho}$ sources the expansion of the universe (in Friedmann equations).

On perturbations with scales $k < k_J$, gravity dominates, ω is imaginary and we have an exponentially increasing mode (gravitational collapse), whereas for scales $k > k_J$, quantum pressure dominates, the solutions in Eq. (3.38) are just waves, and collapse is halted. This last regime is responsible for the small scale power suppression of ULDM.

3.3 ULDM constraints

The main features of ULDM we described in the previous section were used to put constraints on the allowed mass and abundance of ULDM.

On the cosmology side, the ULDM suppression of small scales can be tested by looking at Lyman-alpha forest data. From it, ULDM with $m \lesssim 10^{-21}$ eV is in tension with data [Ir17; Kob+17; Arm+17; Leo+19]. Lyman-alpha forests are also sensitive to mixed ULDM and CDM scenario. Denoting with α_χ the cosmological ratio between ULDM and total DM density, Ref. [Kob+17] found $\alpha_\chi \lesssim 0.2$ for $10^{-23} \lesssim m/\text{eV} \lesssim 10^{-21}$. Suppression of small scales can be probed also with CMB and LSS; Ref. [Lag+22] constrained ULDM fractions in the very light regime, in particular $\alpha_\chi \lesssim 0.3$ for $m \sim 10^{-25}$ eV and $\alpha_\chi \lesssim 0.1$ for $10^{-32} \lesssim m/\text{eV} \lesssim 10^{-26}$.

For $m \lesssim 10^{-21}$ eV, the ULDM core prediction is in tension with observations of the rotation curves of low-surface-brightness galaxies, if a single ULDM species makes up the full DM budget [Bar+18; BBS22; Ber+18], and if solitons are as massive as the relation discussed in Ref. [Sch+14] predicts (in Sec. 6.2, we will investigate such relation, showing we substantially agree with Ref. [Sch+14]).

Dynamical heating-based analyses [Zim+24; TCB25] give comparable constraints to cosmology, $m \gtrsim 10^{-21}$ eV. Note the strong constraints from ultrafaint dwarf galaxies from Ref. [DK22] (see also Ref. [MN19; CSC21]), $m \gtrsim 10^{-19}$; if confirmed to be reliable, it would be the strongest constraint on ULDM to date.

Many other studies have attempted to constrain ULDM using cosmological or astrophysical probes. While we do not aim to provide a comprehensive list, we note that the majority of these works are consistent with the bound $m \gtrsim 10^{-21}$ eV.

To summarise, observational constraints on the magnitude of such effects bounds the particle mass of a ULDM candidate m that comprises all of DM to satisfy $m \gtrsim 10^{-21}$ eV. Among such constraints, this thesis adds the constraint $m \lesssim 5 \times 10^{-25}$ eV via star dynamical heating in dwarf galaxies, discussed in Sec. 6.1, and proposes a method to constrain $\alpha_\chi \lesssim 0.15$ fractions in the mass range $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$ via gravitational lensing time delays, discussed in Ch. 5.

Chapter 4

The Mass Sheet Degeneracy in Time Delay Cosmography

As we already explained in Sec. 2.2.1, the MSD is a well-known mathematical degeneracy arising in lensing mass model reconstructions, which critically affects time-delay predictions. Since time delays can be used to measure H_0 [Ref64; Koc02b; Koc06], it is important to understand to which extent such a degeneracy affects this measurement.

In particular, given how much the accounting of the internal MSD affected the systematic error in the H_0 measurement in Ref. [Bir+20] (see Sec. 1.3), it is important to understand what are the physical realizations of such a mathematical degeneracy. These physical realizations can be summarized in the aforementioned internal MSD, and external MSD. In the former, the mass sheet arises from mass distribution at the same redshift of the main deflector; in the latter, the mass sheet comes from the mass along the LOS. In this chapter, we discuss the effects of these different types of MSD as presented in Refs. [BT21b; Teo+22; TB23].

In Sec. 4.1, we discuss the possibility that an internal MSD comes from a cored dark matter profile in the lensing galaxy halo itself, the type of MSD which affected the TDCOSMO result the most [BT21b]; in Sec. 4.2, we discuss the internal MSD induced by the dark matter halo of the host group, and an interesting discrete degeneracy happening for this setting [TB23]; in Sec. 4.3, we finally discuss the LSS effects as external MSD in the interesting context of multiple source systems [Teo+22].

4.1 Internal lens component

Consider a lensing reconstruction model $\kappa_0(\theta)$ for the convergence of the lens. An internal MSD coming from unaccounted core components in the DM halo of the lensing galaxy, which we will call core-MSD, can be constructed from $\kappa_0(\theta)$ by adding the core component $\kappa_c(\theta)$ while rescaling the original model:

$$\kappa(\theta) = \kappa_c(\theta) + (1 - \kappa_c(\theta_E))\kappa_0(\theta). \quad (4.1)$$

Here θ_E is defined by $\alpha_0(\theta_E) = \theta_E$, where $\alpha_0(\theta)$ is the deflection angle due to $\kappa_0(\theta)$. It is thus the Einstein angle of the model $\kappa_0(\theta)$. At the same time, the source plane coordinates are rescaled as $\beta = (1 - \kappa_c(\theta_E))\beta_0$. In other words, we define a core-MSD as a MSD as presented in Sec. 2.2.1 with $\lambda = 1 - \kappa_c(\theta_E)$. On angular scales $\theta \gg \theta_E$ it is assumed that $\kappa_c(\theta) \rightarrow 0$.

Eq. (4.1) is an approximate MSD if $\kappa_c(\theta)$ is nearly constant up to $|\theta|$ that is sufficiently larger than $|\theta_E|$. To be quantitative, we can define the correction δ_E via:

$$\alpha(\theta_E) =: \theta_E (1 + \delta_E), \quad (4.2)$$

where $\alpha(\theta)$ is the deflection angle of the full model. δ_E quantifies the relative imaging error in the vicinity of $\theta \approx \theta_E$, the angular range where lensing analyses have the most constraining power. For simplicity, in this estimate we assume the system to be spherically symmetric, so that $\alpha(\theta) = 2\theta \int_0^1 dz z \kappa(z\theta)$ (from Eq. (2.37)). Using Eq. (4.1) we then have

$$\delta_E = 2 \int_0^1 dz z (\kappa_c(z\theta_E) - \kappa_c(\theta_E)) = \frac{\alpha_c(\theta_E)}{\theta_E} - \kappa_c(\theta_E). \quad (4.3)$$

While this error estimate was given for a spherical lens, it gives a good approximation of the imaging error also for the non-symmetric systems arising in realistic analyses, as we will verify when discussing a core-MSD given by ULDM in Ch. 5.

Consider the possibility that a lens galaxy harbours a core component, leading to a true convergence profile resembling Eq. (4.1) with $\kappa_c(\theta_E) > 0$. In this case, both the null model $\kappa_0(\theta)$ and the core-MSD model $\kappa(\theta)$ would give a good description of the imaging data. However, the true value of H_0 would differ from the inferred value in the null model by:

$$\frac{H_{0,\text{inferred}} - H_{0,\text{true}}}{H_{0,\text{true}}} \equiv \frac{\delta H_0}{H_0} \approx \kappa_c(\theta_E). \quad (4.4)$$

A good core-MSD able to give a order 10% shift in H_0 will then have to come from a core component whose convergence κ_c satisfies Eq. (4.4) and which produces an imaging error, possibly quantified by δ_E in Eq. (4.3), which is below the sensitivity of imaging capabilities. As we will show in Ch. 5, ULDM in the very light mass regime (although while not comprising the full DM budget) can form cores with these requirements.

4.1.1 Core-MSD and stellar kinematics

Stellar kinematics measurements can in principle break the MSD, and are the limiting observational factor to a core-MSD shift of H_0 discussed previously. The basic observable is the luminosity-weighted velocity dispersion along the line of sight, σ_{los} , given by Eq. (1.93). The core-MSD model enters Eq. (1.93) via the mass profile, $M(r) = M_c(r) + (1 - \kappa_c)M_0(r)$, where $M_0(r)$ comes from the null model and $M_c(r)$ from the core. The dispersion of the full model is related to that of the null model, $\sigma_{\text{los},0}^2$, via

$$\frac{\sigma_{\text{los}}^2}{\sigma_{\text{los},0}^2} = 1 - \kappa_c (1 - \delta_c), \quad \delta_c := \frac{1}{\kappa_c} \frac{\sigma_c^2}{\sigma_{\text{los},0}^2}, \quad (4.5)$$

where σ_c^2 is the velocity dispersion due to the core itself. In general, all of $\sigma_{\text{los},0}^2$, σ_c^2 , σ_{los}^2 , and δ_c depend on the measurement point θ . In Eq. (4.5), the term δ_c parametrises the deviation from the perfect MSD limit. It becomes small for $\theta_c \gg \theta$, but may be quantitatively relevant once we consider a finite core, and once kinematics data probing θ not much smaller than θ_c is used.

To see this, consider an isothermal PL profile (Eq. (2.43) with $\gamma = 2$) for the null model, for which $M_0(\theta) = 2\Sigma_{\text{cr}} D_l^2 \theta_E \theta$. We also take the isotropic limit of the K in Eq. (1.94), and consider the Hernquist profile for the luminosity density, $n_*(r) = n_0 r_*^4 / [2\pi r(r + r_*)^3]$. The parameter r_* is related to the commonly used effective radius r_e via $r_e \approx 1.8r_*$ [Her90]. With these simplifications, and using the core parametrization

$$\rho(r) = \frac{\rho_0}{(1 + a^2 r^2)^{2b}}, \quad (4.6)$$

(such a form mimicks the soliton profile in an external background, see Ch. 5), we have

$$\delta_c(\theta) = \frac{\pi\theta^2}{3\theta_E\theta_c} f\left(\frac{\theta_c}{\theta}, \frac{\theta_*}{\theta}\right) \approx 0.31 \left(\frac{10''}{\theta_c}\right) \left(\frac{1''}{\theta_E}\right) \left(\frac{\theta}{0.5''}\right)^2 \frac{f\left(\frac{\theta_c}{\theta}, \frac{\theta_*}{\theta}\right)}{f(20, 0.5)}, \quad (4.7)$$

where

$$f(y_c, y_*) = \frac{\frac{2\Gamma(2b)}{\sqrt{\pi}\Gamma(2b-\frac{1}{2})} \int_1^\infty dy \frac{\sqrt{y^2-1}}{(y+y_*)^3} {}_2F_1\left(\frac{3}{2}, 2b, \frac{5}{2}, -\frac{y^2}{y_c^2}\right)}{\int_1^\infty dy \frac{\sqrt{y^2-1}}{y^2(y+y_*)^3}}. \quad (4.8)$$

TDCOSMO-IV [Bir+20], in considering the effect of an “internal MSD”, have assumed in practice the perfect MSD limit $\delta_c = 0$ in their kinematics analysis of TDCOSMO and SLACS systems. The approximation was tested using a mock system with $\theta_e = 0.2''$ and several core radii¹. However, the parametric breaking of the MSD, captured by Eq. (4.7), was not explored for different values of θ_e or the baseline θ_E and system redshifts. As Eq. (4.7) suggests a strong dependence on the kinematics observation point, θ , it is important to check to what extent the MSD limit is expected to hold across different systems.

For TDCOSMO systems [Mil+20; Bir+20], the kinematics constraints were based on a single effective measurement centred on $\theta = 0$ and averaged over an aperture $\mathcal{A}$, weighted by the surface brightness $\Sigma_*(\theta)$. To be more precise, the observationally accessible dispersion σ_{los} is given by Eq. (1.96), which we can rewrite as [Suy+10; BAR16]

$$\langle\sigma_{\text{los}}\rangle^2 = \frac{\int_{\mathcal{A}} d^2\theta [\Sigma_*(\theta) \sigma_{\text{los}}^2(\theta) * \mathcal{P}]}{\int_{\mathcal{A}} d^2\theta [\Sigma_*(\theta) * \mathcal{P}]}, \quad (4.9)$$

where $\mathcal{P}$ is the seeing. It is natural to define

$$\frac{(\langle\sigma_{\text{los}}\rangle)^2}{(\langle\sigma_{\text{los},0}\rangle)^2} = 1 - \kappa_c (1 - \Delta_c), \quad (4.10)$$

$$\Delta_c = \frac{\int_{\mathcal{A}} d\theta \theta \Sigma_*(\theta) \sigma_{\text{los},0}^2(\theta) \delta_c(\theta) * \mathcal{P}}{\int_{\mathcal{A}} d\theta \theta \Sigma_*(\theta) \sigma_{\text{los},0}^2(\theta) * \mathcal{P}}. \quad (4.11)$$

From this expression and the previously quoted results, the correction term Δ_c can be evaluated numerically. It depends on θ_c, θ_* (equivalently θ_e), the aperture $\mathcal{A}$, and the seeing $\mathcal{P}$. The main point to explore is how Δ_c reacts to different values of θ_e and θ_c .

In Fig. 4.1 we plot Δ_c vs. θ_e for different values of θ_c . The null model is defined with $\theta_E = 1''$. The aperture is defined to be a circular region of radius $1''$ (a simplification of the aperture in [Bir+20]). For simplicity we neglect the seeing, setting the Full Width Half Maximum (FWHM) of $\mathcal{P}$ to zero.

TDCOSMO systems typically have $\theta_E \sim 1''$, and from the imaging analysis we know that $\theta_c \gtrsim 5\theta_E$ or so. Some TDCOSMO systems have $\theta_e \sim \theta_E \sim 1''$ (Fig. 16 in [Bir+20]); for such systems, Δ_c can exceed 30%. SLACS systems have even larger values of θ_e , some reaching $\theta_e \sim 2.5\theta_E$, and Fig. 4.1 shows that the MSD limit may be violated at the $\mathcal{O}(1)$ level. The effect should be even more important for SLACS systems with resolved kinematics (see Figs. 15 [Bir+20]). This is manifest, to some extent, in Fig. B3 in [Bir+20]. The calculation in Fig. 4.1 does not include the effect of a finite PSF, velocity anisotropy, lens ellipticity, etc. In Fig. 4.2, we repeat a similar calculation using a full mock system that includes all of these effects, using *lenstronomy* [BAR16; BA18a; Bir+21]. The result of a full computation is compatible with that in Fig. 4.1 numerically to 50% or so.

¹The core toy model in [Bir+20] was different from the core we considered, Eq. (4.6). The kinematics effect is approximately matched between the two models for $\theta_c^{(\text{toy})} \approx 0.5\theta_c^{(\text{here})}$.

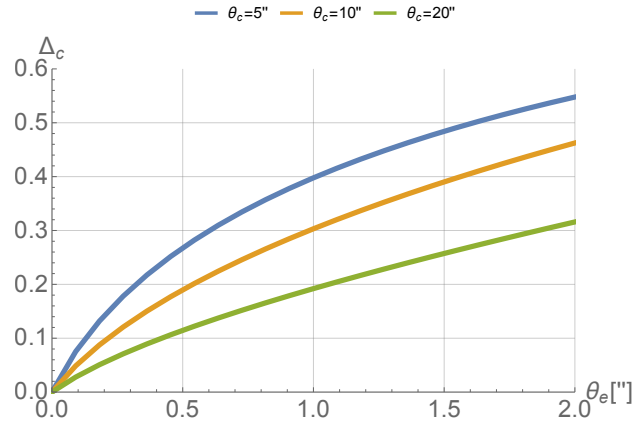


FIGURE 4.1: The finite-core correction Δ_c , modifying the MSD limit in the kinematics analysis (see Eqs. (4.10-4.11) and text). Here we neglect the seeing, the aperture is defined to be a circular region of radius $1''$, and the null model has $\theta_E = 1''$.

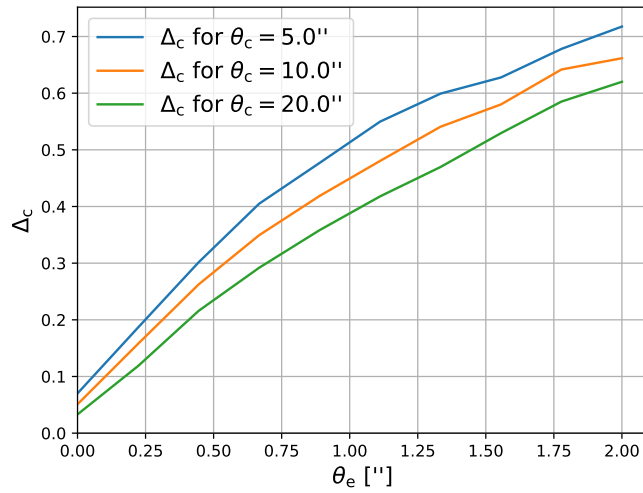


FIGURE 4.2: Plotting σ dispersion, for a mock lens and the influence of a core parametrized via Eq. (4.6), using `lenstronomy`. Differently from Fig. 4.1, here we use realistic seeing, apertures, and use a Osikpov-Merrit anisotropy profile, Eq. (1.86).

If a real physical core component is behind the lensing H_0 tension, then the kinematics constraints must be considered with care, because the MSD limit could introduce large systematic errors. In general, the breaking of the MSD manifests in a *smaller deviation* of $\langle\sigma_{\text{los}}\rangle$ from the null model: instead of $\delta\langle\sigma_{\text{los}}\rangle / \langle\sigma_{\text{los}}\rangle = -0.5\kappa_c$ we have $\delta\langle\sigma_{\text{los}}\rangle / \langle\sigma_{\text{los}}\rangle = -0.5\kappa_c(1 - \Delta_c)$, with $\Delta_c > 0$. This calls into question the kinematics analysis of some TDCOSMO systems and certainly of resolved SLACS systems in [Bir+20].

The point that assuming the perfect MSD limit is not conservative was first made in Ref. [BT21b]. After the release of that work, TDCOSMO implemented a finite-core parametrization of the core-MSD in their modelling, see e.g. Ref. [Sha+23].

4.2 Group dark matter halo: the flexion degeneracy

On scales in between cosmology and lens internal structure, massive galaxies like the lenses of [Rus+20; Bir+19; Sha+20; Che+19; Won+20; Mil+20] are often members of a group², and the dark matter halo of a host group may act to some extent as an internal MSD. This host group dark matter halo induced MSD is yet another possible source of systematic bias in time-delay cosmography. Ref. [Wil+16; Wil+17] studied the impact of host and LOS groups on lensing systems. Among the systems considered, PG1115+080, RXJ1131-1231, HE0435-1223, and WFI2033-4723 featured in the H_0 campaign of [Mil+20]³. Interestingly, PG1115+080 yields a high central value of H_0 (81^{+8}_{-7} km/s/Mpc) [Mil+20]; at the same time, this lens resides in a massive group, inducing convergence $\kappa_g \approx 0.2$ [Wil+17], where we denoted with κ_g the convergence of the host group at the main lens location. The group convergence goes directly into the inference of H_0 , via $\delta H_0 / H_0^{\text{truth}} \approx -\delta\kappa_g$, where $\delta\kappa_g = \kappa_g^{\text{truth}} - \kappa_g^{\text{model}}$ is any error in the model determination of κ_g , and $\delta H_0 = H_0^{\text{truth}} - H_0^{\text{model}}$. We do not know if the lensing results for PG1115+080 are indeed biased by its group modelling, but clearly, it is important to understand to what accuracy can lensing analyses determine κ_g .

With the example of PG1115+080 in mind, we here show numerical mock data analysis of a gravitational lens in a NFW halo, targeted at the study of such a group-induced MSD. We find that the posterior likelihood of the lensing reconstruction problem exhibits three disjoint minima in Fig. 4.4,4.5. These minima reflect an approximate discrete degeneracy, related to the transformation properties of a dominant flexion term under coordinate rotations around the primary lens. This degeneracy affects the prediction for κ_g , thereby biasing the H_0 prediction, if time delays are used for measuring H_0 . In particular, false minima in the posterior landscape have an H_0 inference which is biased up.

We describe in Sec. 4.2.1 how the host group can be seen as a core-MSD discussed in Sec. 4.1 and how the discrete flexion degeneracy arises; we describe our mock setup, based on the PG1115+080 system, in Sec. 4.2.2 and describe the main results in Sec. 4.2.3; we explain why the flexion degeneracy biases up H_0 in Sec. 4.2.4. We describe the kinematics constraints for the host group in Sec. 4.2.5 and make further discussions of our results in Sec. 4.2.6,4.2.7,4.2.8 and summarize in Sec. 4.2.9.

4.2.1 Host group as a core-MSD

To show the influence of a displaced group in a gravitational lens system and explain the discrete degeneracy we find, it is convenient to adopt complex notation for 2D angle vectors on the sky [SK95; SE08], defining, e.g.,

$$\theta = \theta_1 + i\theta_2, \quad \beta = \beta_1 + i\beta_2, \quad (4.12)$$

²Ref. [Wil+16] finds that half of their sample of 26 galaxy lenses can be associated with a group.

³At least two additional systems from [Mil+20] (SDSS1206 [Bir+19] and DES J0408 [BG+20]), do not feature in [Wil+16; Wil+17], but are known to also be associated with host groups. Thus at least 6 out of the 7 lensed quasars from [Mil+20] involve a group.

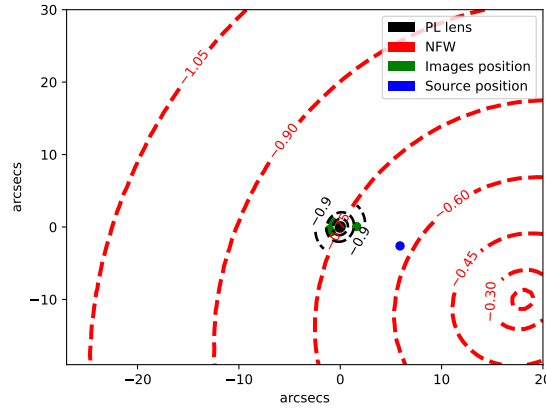


FIGURE 4.3: Illustration of the mock setup. Contours show $\log_{10}$ of the convergence for the primary lens and the NFW halo in black and red, respectively.

and so on. We show more details of the complex notation in App. D.2. With this formalism, the lens equation can be written as

$$\beta = \theta - \alpha_1(\theta) - \Delta\alpha(\theta) - \kappa_{\text{ext}}\theta - \gamma_{\text{ext}}\theta^*. \quad (4.13)$$

Here, $\alpha_1(\theta)$ is the deflection angle due to the primary lens, $\Delta\alpha(\theta)$ is the deflection due to the host group or cluster, and β is the source position. The external convergence and shear, κ_{ext} and γ_{ext} , contain a combination of different LOS contributions, see Sec. 2.4.

For a group located at center of mass position $e^{i\phi}h$, with $h \gg |\theta|$, $\Delta\alpha$ can be expanded as a power series in θ (see Ref. [SE08] and App. D.2):

$$\Delta\alpha(\theta) = \Delta\beta + \Delta\kappa\theta + \Delta\gamma\theta^* + \frac{1}{4}F^*\theta^2 + \frac{1}{2}F\theta\theta^* + \frac{1}{4}G\theta^{*2} + \dots \quad (4.14)$$

In the case of an axisymmetric group halo profile, it is possible to decompose the expansion coefficients as

$$\Delta\beta = \Delta\beta_0 e^{i\phi}, \quad \Delta\gamma = \Delta\gamma_0 e^{2i\phi}, \quad F = F_0 e^{i\phi}, \quad G = G_0 e^{3i\phi}, \quad (4.15)$$

where $\Delta\beta_0, \Delta\gamma_0, F_0, G_0$ are real numbers that depend on h but not on ϕ . Note that $\Delta\kappa$ is independent on ϕ .

Of the expansion coefficients in Eq. (4.14), $\Delta\beta$ is degenerate with β , which is a free parameter in the modeling, and has no effect on time delays. $\Delta\kappa$ is degenerate with κ_{ext} , a free parameter.⁴ Varied along the MSD, the combination $\Delta\kappa + \kappa_{\text{ext}}$ is invisible to imaging, but affects time delays. $\Delta\gamma$ is degenerate with γ_{ext} , a free parameter. Thus, as far as imaging data is concerned, the leading order effect that constrains the group comes from the MSD-invariant combinations $F/(1 - \Delta\kappa - \kappa_{\text{ext}})$ and $G/(1 - \Delta\kappa - \kappa_{\text{ext}})$, related to what is known in the literature as reduced flexion [GB05; Bac+06].

The most important point to notice in the expansion Eq. (4.15) is the phase degeneracy of the G term. This degeneracy means that models of the halo in which the direction to the group's center is changed by $\phi \rightarrow \phi \pm 2\pi/3$ yield identical G terms. Although the rotated models do not reproduce the truth value of the F term, the G degeneracy can produce isolated local minima in the posterior likelihood. MCMC algorithms which are not suited for multimodal posterior scans may remain stuck in a local minimum which does not contain the true values. We call such minima false minima. In these minima, the inference of H_0 will be biased up, as we show in Sec. 4.2.3.

⁴A cosmological prior can sometimes be imposed on κ_{ext} , by comparing the system's field to N-body simulations [Hil+09].

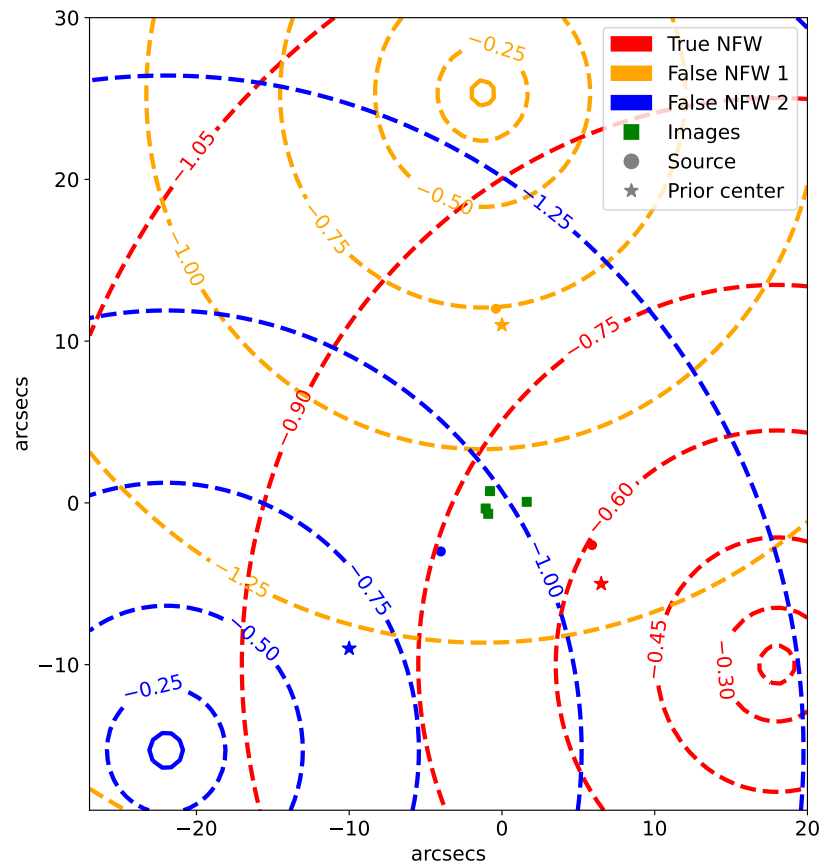


FIGURE 4.4: Representation of the three log-posterior minima, when attempting a Bayesian analysis of the mock system represented in Fig. 4.3.

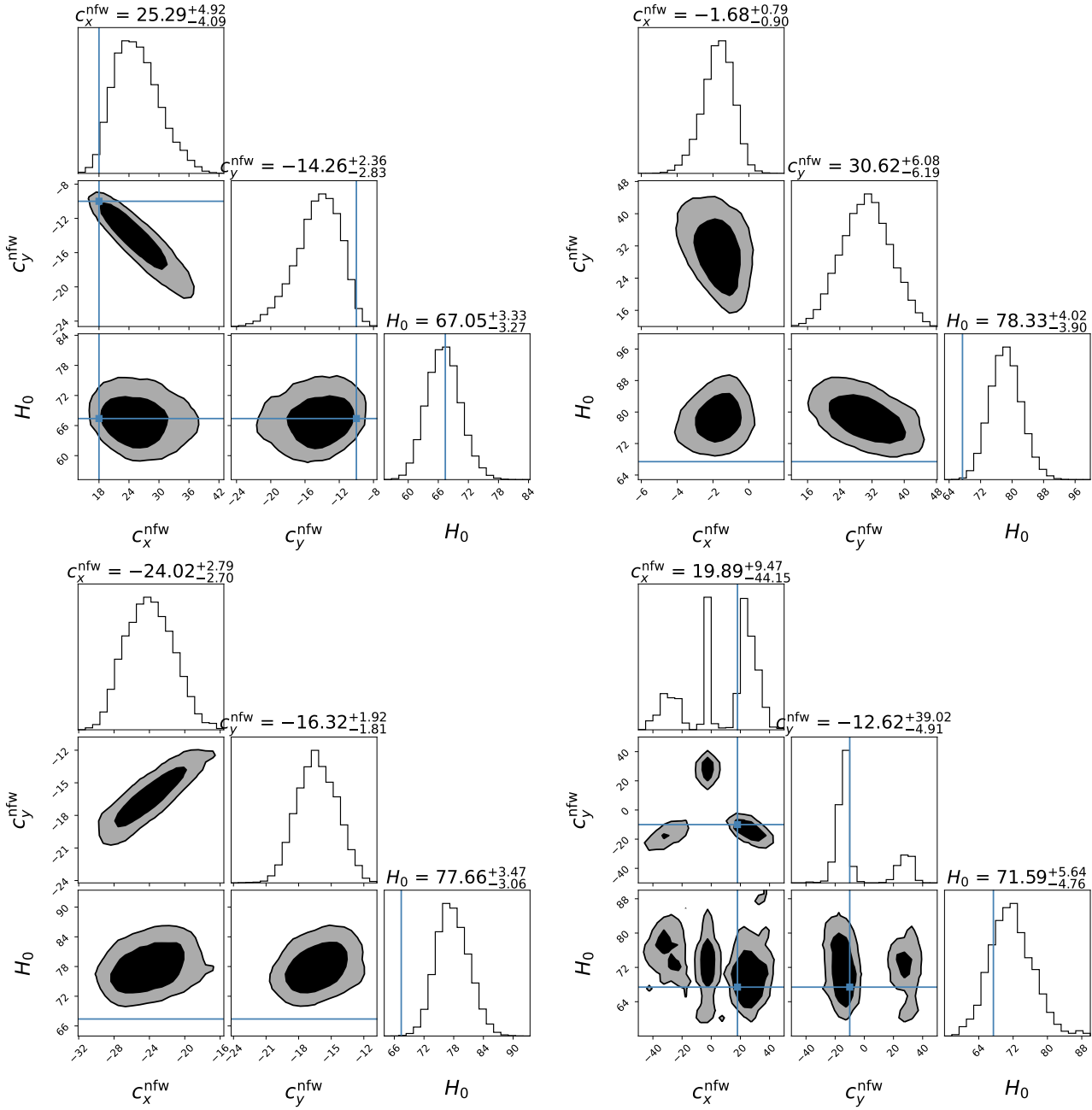


FIGURE 4.5: Corner plots of Bayesian parameter inference of the mock system represented in Fig. 4.3. Upper-left panel is the corner plot likelihood corresponding to the truth parameters; upper-right and lower-left relate to the false minima instead. Lower-right panel shows a parameter inference with zeus [KBP21], particularly suited for scanning multimodal posteriors. One can see the full posterior landscape, with the false minima included.

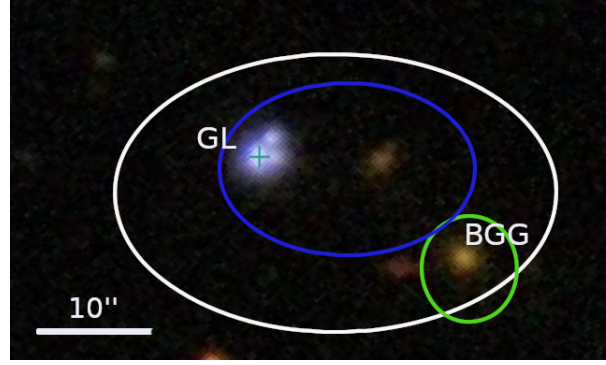


FIGURE 4.6: The field of PG1115+080 (image from SIMBAD [Wen+00]). The primary lens, GL, is marked by a cross, surrounded by the quasar images. White ellipse shows the 68% CL prior derived in [Wil+16] and used in [Che+19] to constrain the group’s center of mass. Green and blue ellipses give a rough illustration of the X-ray emission associated to the group, found in [Gra+04] and [Fas+08], respectively, using different subtraction schemes for the quasar emission.

4.2.2 Illustration with mock data: mock setup

The host group is often modelled explicitly if the system is known to reside in a group (see e.g. [Mil+20; Mom+06; Wil+16; Wil+17]). We now study such modelling using mock data. Our implementation is based on the package `lenstronomy` [BA18b; BAR15; Bir+21].

We chose PG1115+080 as a reference object to guide our study. Ref. [Mil+20] inferred $H_0 = 81^{+8}_{-7}$ km/s/Mpc from this system, quite high compared to the CMB value [Agh+20]. At the same time, the system is known to reside in a group [Wil+17] with $\sigma_{\text{los}} = 390^{+50}_{-60}$ km/s (an earlier study found $\sigma_{\text{los}} = 440^{+90}_{-80}$ km/s [Mom+06]), estimated to induce $\kappa_g \sim 0.2$. The group’s projected center of mass is not far from the primary lens (h of the order of $10''$). This suggests that flexion terms are probably not negligible for this system, distorting the image on scales $\Delta\theta \sim \kappa_g \theta_E^2 / h \sim 0.01''$. An image of the field is shown in Fig. 4.6.

We consider the following mock setup. For the primary lens, we consider an elliptic power-law density profile with 3D slope $\gamma = 2.17$, Einstein angle $\theta_E = 1.08''$, and ellipticity parameters $e_1 = -0.2$, $e_2 = 0.05$, corresponding to $q = (1 - e)/(1 + e) \approx 0.66$, where $e = \sqrt{e_1^2 + e_2^2}$. For the group, we consider an elliptic NFW profile [JS02; VfyG17] with $e_1 = -0.07$, $e_2 = 0.03$, compatible with the findings of Ref. [Sch+21]. Fig. 4.3 illustrates the setup, including the truth position of the group halo center of mass. This setup could mimic PG1115+080 (Fig. 4.6) if the BGG – or the X-ray blob found by Ref. [Gra+04] – happens to indicate the group’s center of mass position.

Following Ref. [Che+19], we choose our inference pipeline to include only a simple spherical group model, ignoring group halo ellipticity in the modelling.

External priors: tracer galaxy kinematics and theoretical input from N-body simulations

As in Ref. [Che+19], we include external priors on the group halo, dictated by cosmological N-body simulations and by kinematics data. We defer most of the details to Sec. 4.2.5. However, we would like to point out a potential inaccuracy in the kinematics analyses of some previous works.

Galaxy groups are often assumed to follow the NFW density profile [NFW97],

$$\rho(r) = \frac{\rho_s R_s^3}{r(r + R_s)^2}. \quad (4.16)$$

In terms of the parameters ρ_s and R_s , the LOSVD of tracer galaxies (with number density assumed to follow the same profile as the dark matter density) can be expressed as

$$\sigma_{\text{los}}^2(\theta) = G\rho_s R_s^2 f\left(\frac{\theta}{\theta_s}\right), \quad (4.17)$$

where $f(a)$ is a dimensionless function derived in Sec. 4.2.5.

While Eq. (4.17) (averaged as needed within some aperture cut of the observations) gives the correct translation between LOSVD data and the NFW model parameters, Ref. [Che+19] (after [KG00; Won+11]) considered a different expression as a proxy for the LOSVD data:

$$\bar{\sigma}^2 = \frac{GM_{\text{vir}}}{3R_{\text{vir}}}, \quad (4.18)$$

where R_{vir} and $M_{\text{vir}} = M(R_{\text{vir}})$ are the virial radius and virial mass, and $c_{\text{vir}} = R_{\text{vir}}/R_s$ is the NFW concentration parameter. In Sec. 4.2.5 we show that identifying the quantity $\bar{\sigma}^2$ with the observable σ_{los}^2 introduces an error of up to $\sim 20\%$ (the precise error depends on the analysis aperture).

One reason for introducing the auxiliary quantities R_{vir} and M_{vir} , is that N-body simulations provide theoretically-motivated priors that are often presented in terms of these quantities [ENS01; MDB08]. These cosmological priors are, however, quantitatively and conceptually decoupled from the kinematics data interpretation. Instead, as we review in Sec. 4.2.5, the cosmological priors dictate a certain redshift-dependent relation between the NFW parameters ρ_0 and R_s . There is no obstacle to implement this theoretical prior while still maintaining the correct kinematical expression, Eq. (4.17).

In our main implementation of the MCMC, we set the standard deviation for σ_{los} to 120 km/s. This doubles the nominal uncertainty quoted by [Wil+17] for the group of PG1115+080, but we believe that such cautionary procedure is reasonable. To be clear, we are not suggesting to doubt the observational LOSVD from [Wil+17]. Rather, the uncertainties we worry about concern the theoretical interpretation within simplified halo models. The systematic error due to using $\bar{\sigma}$ instead of σ_{los} , as done in [Che+19], is of the order of $\delta\sigma_{\text{los}}^2/\sigma_{\text{los}}^2 \sim 20\%$, so $\delta\sigma_{\text{los}}/\sigma_{\text{los}} \sim 10\%$, or $\delta\sigma_{\text{los}} \sim 40$ km/s, quite comparable to the “bare” observational uncertainty. There are additional plausible errors: the group is likely to be aspherical [Sch+21], the velocity distribution need not be isotropic, and the group may not be fully virialised. Each of these could cause systematic shifts of tens of percent in the kinematics interpretation.

For completeness, in Sec. 4.2.5 we check how a LOSVD uncertainty of 60 km/s changes the results. We find the difference to be quantitatively insignificant for our main results.

4.2.3 Group halo mock: results

First, to obtain a global view of the posterior “landscape”, we run the zeus MCMC algorithm [KB21; KBP21], which is designed to cope with multiple likelihood minima. We show the main result in the **top-left panel** of Fig. 4.7. Notice the 3 disjoint minima in the posterior likelihood in the $c_x^{\text{nfw}} - c_y^{\text{nfw}}$ panel, where $c_{x,y}^{\text{nfw}}$ are the angular coordinates of the halo center of mass. The origin of this threefold degeneracy is the $\phi \rightarrow \phi \pm 2\pi/3$ degeneracy of the flexion G term. This run has no prior on c_x^{nfw} and c_y^{nfw} .

The local minima could trap an MCMC, if the scanning algorithm is not suited to probe multimodal posteriors. To see this, we repeat the analysis, this time using the emcee algorithm [FM+13]. The results are shown in the **top-right** and **bottom panels** of Fig. 4.7. Indeed, emcee tends to discover only one of the local minima, missing the others. In the **top-right** and **bottom-right panels**, emcee converges on a biased group halo position. The biased local minima yield an H_0 posterior that is on the high side,

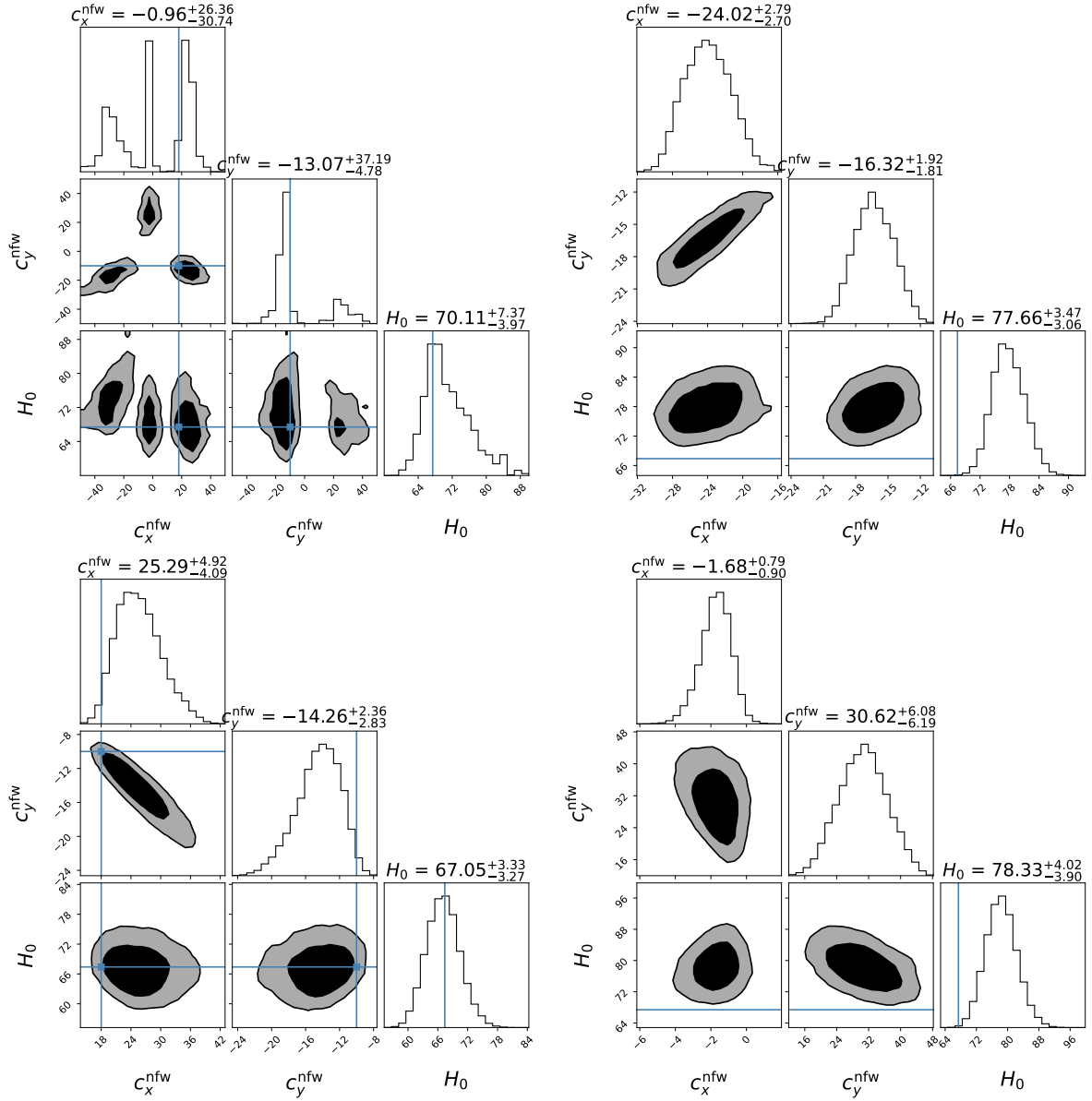


FIGURE 4.7: Mock analysis. Truth set-up: elliptic group located at $c_x^{\text{nf}} = 18''$, $c_y^{\text{nf}} = -10''$. **Top left panel:** zeus run, exposing the global likelihood landscape. **Other panels:** emcee runs, falling into different local minima. The emcee runs are initiated with different priors for the position of the group center, keeping the same Gaussian standard deviation of $16''$. See also Fig. 4.8 .

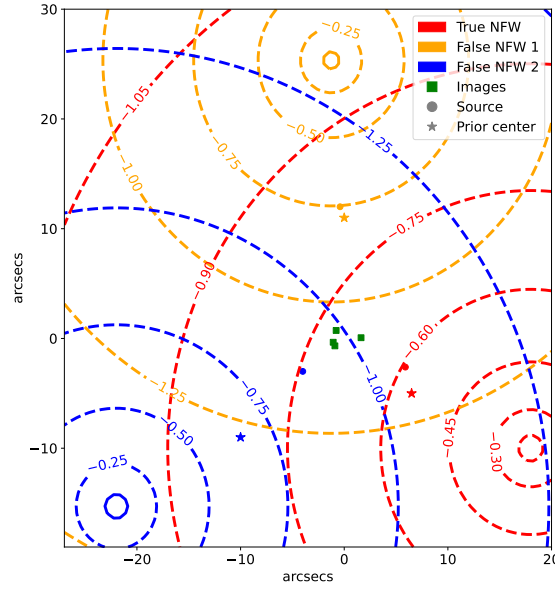


FIGURE 4.8: Illustration of the three posterior likelihood minima probed by the MCMC inference. Contours show $\log_{10}$ convergence isocontours, for the 3 group halo solutions (red, orange, and blue, respectively). Dots and stars highlight the inferred source position and the center of the group prior, respectively. Green squares denote the position of the images. The parameter values used in the plot are read from the `emcee` run best fit, see Fig. 4.7 .

pulling H_0 above its truth value by $\sim 3\sigma$ and $\sim 1.8\sigma$, respectively. The convergence pattern associated to the three minima is shown in Fig. 4.8.

Performing more MCMC runs, we find that `emcee` consistently converges into just one of the minima, missing the others. The choice of the minimum found by `emcee` mostly depends on the initial position of the MCMC walkers in the $c_x^{\text{nfw}} - c_y^{\text{nfw}}$ parameter space. In our runs, the initial walker allocation is guided by a Gaussian prior with a similar 68% confidence level radius as that used in [Che+19] (see Fig. 4.6), and with centers shown by star symbols in Fig. 4.8.

After performing a series of numerical trials we emphasize that, at least for the given, rather broad (but realistic) width of the prior, the starting point of the MCMC walkers appears to be a more significant factor in determining which of the three minima traps the chain, than the prior center itself. These considerations coincide if the MCMC walkers are initiated at the prior center.

Given the above discussion, we can make a rough estimate of the probability of the MCMC to fall into a displaced minimum, by matching it with the probability of the group center prior to be nearer a false minimum than the truth one. In Sec. 4.2.7 we estimate this probability from mock realizations of samples of tracer galaxies. The result depends on the analyses aperture, the number of galaxies in a sample, and the underlying group profile. For 13 members in the NFW profile, with an aperture of $\sim 10R_s$ (i.e., around 3 virial radii), the false minimum probability we find is $\sim 10\%$.

We comment that X-ray data may help to pinpoint the centroid of a massive group, resolving the three-fold degeneracy. For example, Ref. [Gra+04] found an X-ray blob that appears to be centered around the BGG of the host group of PG1115+080 (green contour in Fig. 4.6). If the X-ray emission can be associated with the group's centroid, it may provide a narrower prior than that derived from tracer galaxies. The X-ray analysis may be complicated by blending with the lensed quasar: using a different method to

mask the quasar, Ref. [Fas+08] found a shifted, more extended, and brighter group emission. Even so, it seems plausible that X-ray data could help narrowing the group's centroid prior⁵.

It is natural to ask whether the discrete modeling bias tends to come along with an anomalously large external shear estimate. While this could indeed happen, and might serve as a useful “alarm bell” if the H_0 bias is larger than 10% or so, our numerical examples also show false minima solutions with acceptable values of external shear. For example, Fig. 4.10 shows a false minimum solution with $\sim 15\%$ bias in H_0 but with external shear values that are compatible with those found in [Che+19] for PG1115+080 (see Figs. 7,8 there).

Finally, we emphasize again that PG1115+080 was selected to guide our mock specifically because of its massive, near-by group association. This set-up, while we believe it deserves study on its own right, may be an un-representative outlier among lensed quasar systems, and it is therefore natural to ask just how uncommon it is. A detailed analysis of the fraction of systems that may exhibit flexion degeneracy is beyond the scope of this work, but in Sec. 4.2.8 we attempt a crude estimate using the data from [Wil+17]. Our results suggest that detectable flexion distortion from LOS groups (including, not limited to, the host of the primary lens) may affect $\sim 10\% - 30\%$ of lensed quasar systems.

4.2.4 Why is H_0 biased up?

As we explained, the three-fold approximate degeneracy manifest in Fig. 4.5 is due to the behavior under rotations of the G flexion. The F flexion, however, behaves like a vector. Hence, when the inference falls into a wrong minimum, it attempts to minimize the difference between the truth F deflection and the wrong-minimum inference value of F . This can be achieved by keeping G_{eff} the same, but reducing F_{eff} as much as possible, where

$$F_{\text{eff}} = \frac{F_0}{1 - \Delta\kappa}, \quad G_{\text{eff}} = \frac{G_0}{1 - \Delta\kappa}, \quad (4.19)$$

are the reduced flexion terms (for clarity, here we omit the cosmological term κ_{ext}). To see this point, denote the reduced flexion of the inference model by F_{eff} , and denote the truth flexion by $F_{\text{eff}}^{(\text{truth})}$ ⁶. We expect the MCMC to minimize

$$\left| F_{\text{eff}}^{(\text{truth})} - F_{\text{eff}} e^{\frac{2in\pi}{3}} \right| = \left| F_{\text{eff}}^{(\text{truth})} \right| \sqrt{1 - 2 \frac{F_{\text{eff}}}{F_{\text{eff}}^{(\text{truth})}} \cos\left(\frac{2n\pi}{3}\right) + \left(\frac{F_{\text{eff}}}{F_{\text{eff}}^{(\text{truth})}}\right)^2}, \quad (4.20)$$

with $n = \pm 1$ selecting the position of the false minimum. Since $\cos(\pm 2\pi/3) = -1/2$, the above expression is minimized for $F_{\text{eff}} = 0$.

Consider the PL model, Eq. (2.41). In this model, we have (see App. D.2.2) $F_{\text{eff}}/G_{\text{eff}} = (\gamma_g - 3)/(\gamma_g + 1)$, where γ_g is the power law exponent associated with the group halo. Therefore, for $1 < \gamma_g < 3$ (the range of interest), decreasing $|F_{\text{eff}}|$ at fixed G_{eff} entails increasing γ_g , while adjusting the other parameters of the model so as to keep G_{eff} constant. Now, we have (see Eq. (D.55)) $G_{\text{eff}} = -\frac{\gamma_g^2 - 1}{3 - \gamma_g} \frac{\Delta\kappa}{h(1 - \Delta\kappa)}$. The γ_g -dependent factor, $\frac{\gamma_g^2 - 1}{3 - \gamma_g}$, increases with increasing γ_g ; to compensate for this and keep G_{eff} constant, the factor $\frac{\Delta\kappa}{h(1 - \Delta\kappa)}$ needs to decrease. For small $\Delta\kappa \ll 1$, this means that near any one of the displaced likelihood minima, the MCMC will attempt to decrease $\Delta\kappa/h$ in comparison to its truth value. Part of

⁵In principle, X-ray luminosity measurements can also constrain the group's velocity dispersion and mass. In practice, however, the limited accuracy of X-ray luminosity measurements [Fas+08], combined with the observed scatter in luminosity-mass or luminosity-dispersion correlations [Lov+21], suggests that the corresponding constraints may not lead to significant improvement over kinematics alone.

⁶The truth and model values of the G term are assumed to approximately coincide, $G_{\text{eff}}^{(\text{truth})} \approx G_{\text{eff}}$.

this adjustment entails decreasing the model value of $\Delta\kappa$, which therefore biases H_0 high, as $\Delta H_0/H_0 \approx -(\Delta\kappa^{(\text{truth})} - \Delta\kappa^{(\text{model})})$.

In Fig. 4.9 we show triangle plots in which the cosmological prior on c_{vir} is not included. We do this exercise in order to investigate the impact of this prior on the results. The main point to notice is that omitting the c_{vir} prior, the bias on H_0 becomes somewhat more pronounced (compare Fig. 4.8, that includes this prior). At the same time, without this prior, the best fit result for R_s in displaced (false) posterior likelihood minima is driven to small values. This point is shown by a comprehensive triangle plot in Fig. 4.10.

In Fig. 4.11 we show triangle plots in which the kinematics prior is enforced with a standard deviation of 60 km/s on σ_{los} . These results can be compared to Fig. 4.8 from the main text, where, as noted in Sec. 4.2.2, the standard deviation on σ_{los} was taken as 120 km/s. We do not find a significant difference.

Fig. 4.12 gives a more complete perspective on the degeneracies and the global structure of the likelihood as exposed by a zeus run.

4.2.5 Kinematics constraints for host group

Here we review constraints on a host group, obtainable by measurements of a sample of galaxy members. We only consider spherical systems with an isotropic velocity distribution. We start with a PL mass and tracer galaxy distribution, and go on to consider the NFW profile.

Power law

We start with the PL profile of Eq. (2.41). The circular velocity is

$$v_{\text{circ}}^2(R) = \frac{GM(R)}{R} = \frac{4\pi G\rho_0(D_l\theta_E)^\gamma}{3-\gamma} R^{2-\gamma}, \quad \rho_0 = \frac{\Sigma_{\text{cr}}}{D_l\theta_E} \frac{3-\gamma}{2\sqrt{\pi}} \frac{\Gamma(\frac{\gamma}{2})}{\Gamma(\frac{\gamma-1}{2})}. \quad (4.21)$$

The circular velocity is not directly measurable. What is measurable is the LOSVD [BT08], Eq. (1.93),

$$\sigma_{\text{los}}^2(R) = \frac{8\pi G\rho_0(D_l\theta_E)^3 R^{3-\gamma}}{(3-\gamma)\Sigma_*(r)} \int_1^\infty \frac{dy}{\sqrt{y^2-1}} \int_y^\infty dx n_*(xr) x^{1-\gamma}, \quad (4.22)$$

where we remind that n_* and Σ_* are the galaxy number density and surface density, respectively. If the galaxy number density is distributed similarly to the mass density in the group, $n_*(r) \propto (r/(D_l\theta_E))^{-\gamma}$, we find (valid for $\frac{3}{2} < \gamma < \frac{5}{2}$):

$$\sigma_{\text{los}}^2(\theta) = \frac{\Gamma^2(\frac{\gamma}{2}) \Gamma(\gamma - \frac{3}{2})}{4\sqrt{\pi} \Gamma^2(\frac{\gamma-1}{2}) \Gamma(\gamma)} \frac{D_s}{D_{\text{ls}}} \theta_E \left(\frac{\theta}{\theta_E}\right)^{2-\gamma}. \quad (4.23)$$

For $\gamma = 2$, $\sigma_{\text{los}}^2 = \frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi} = \frac{v_{\text{circ}}^2}{2}$.

The relation $\sigma_{\text{los}}^2 = \frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi}$ is sometimes adopted by lensing analyses. However, the expressions for σ_{los}^2 depend on a series of simplifying assumptions, including: PL mass distribution; same PL galaxy number distribution; virial state (following Jeans equation); spherical symmetry; isotropic velocity distribution; and to satisfy $\sigma_{\text{los}}^2 = \frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi}$, the specific case of the SIS. We do not expect those assumptions to hold precisely for individual galaxy groups. Lensing analyses would be prudent to assign a larger systematic uncertainty than the “bare” observational uncertainty on σ_{los}^2 .

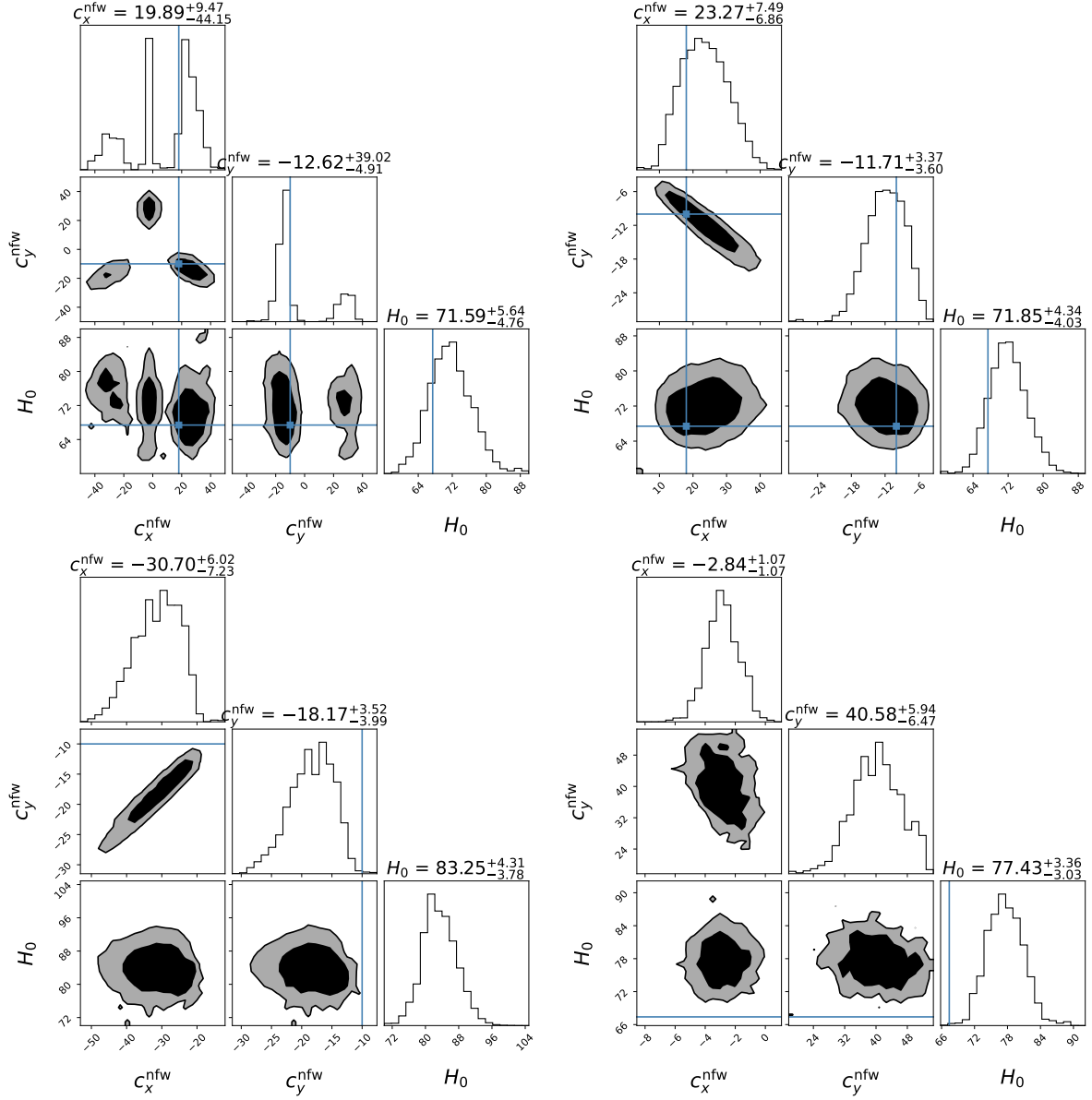


FIGURE 4.9: Mock analysis: *zeus* run (upper-left) and *emcee* runs, all with a kinematics prior but no c_{vir} prior. We note that in false minima fits, the inferred θ_s (not shown here) is driven to very low values $\odot$.

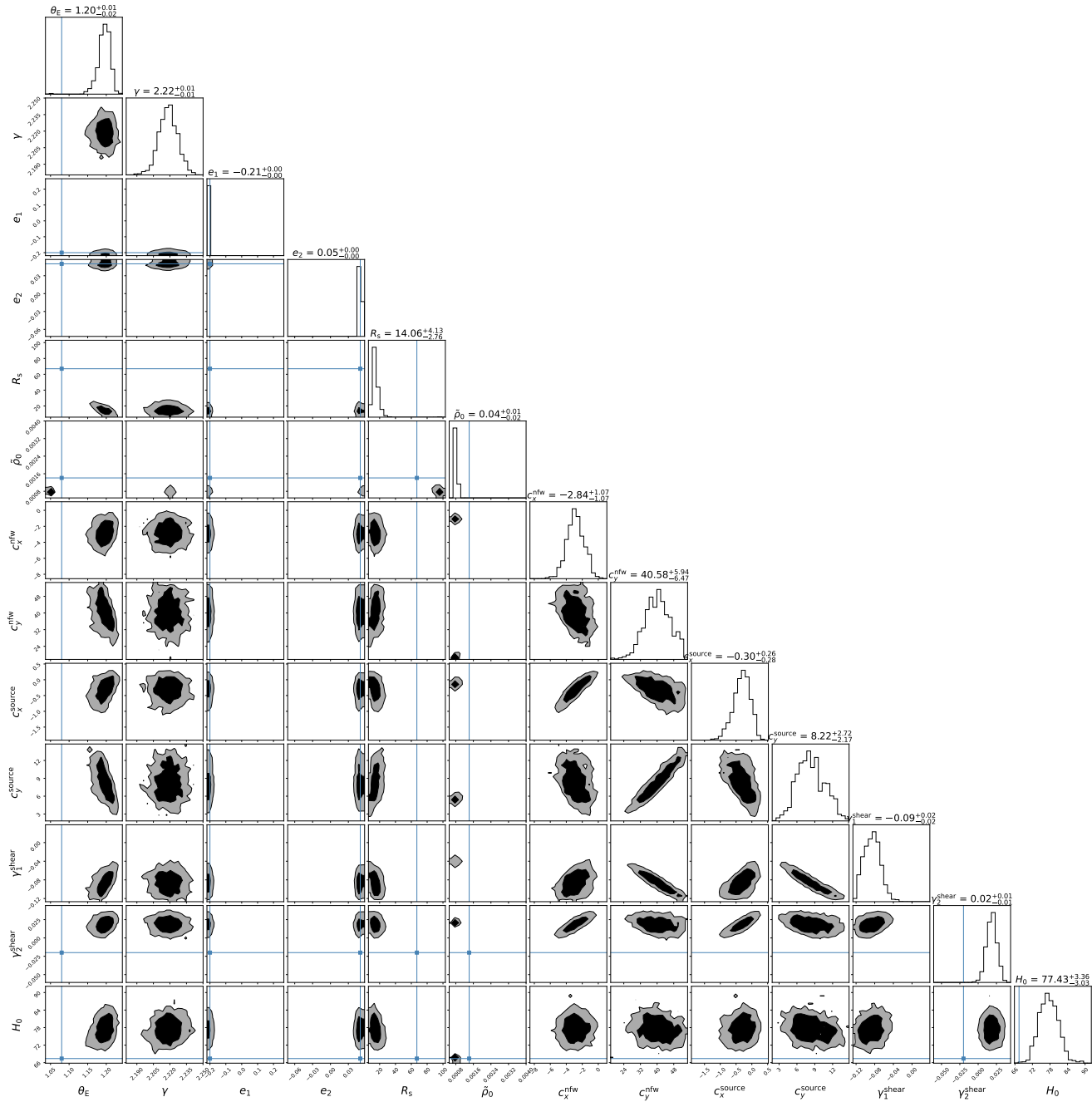


FIGURE 4.10: emcee run, multi-parameter view, corresponding to the run in the bottom-left panel of Fig. 4.9. The absence of a c_{vir} prior causes the inference to pull towards low values of R_s near a displaced minimum. The parameter $\bar{\rho}_0$ is defined as $\bar{\kappa}/\theta_s$ (which is independent on R_s) $\odot$.

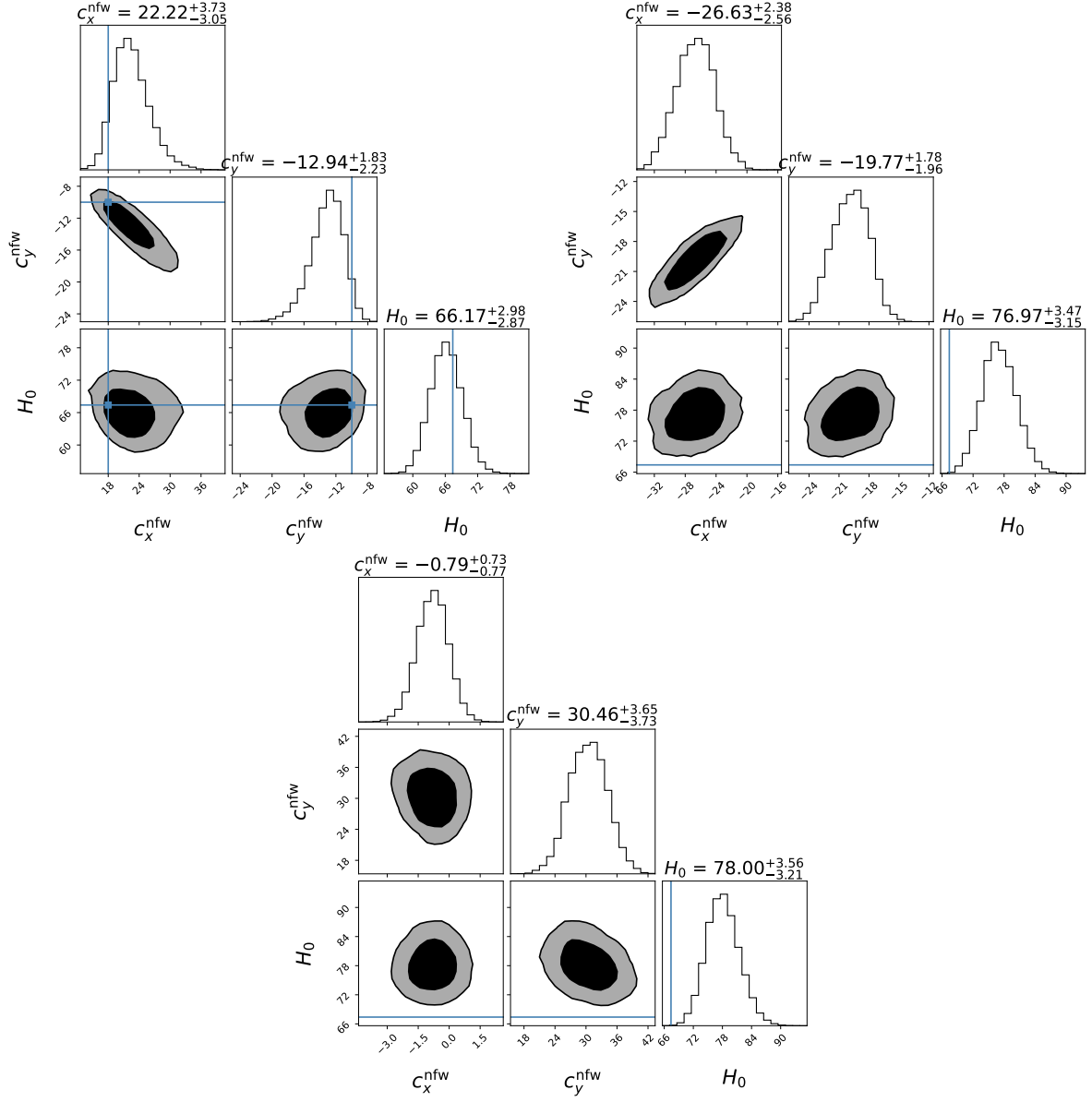


FIGURE 4.11: Mock analysis: *emcee* runs using a standard deviation of 60 km/s on σ_{los} . These results can be compared with Fig. 4.8, in which the uncertainty in σ_{los} was doubled

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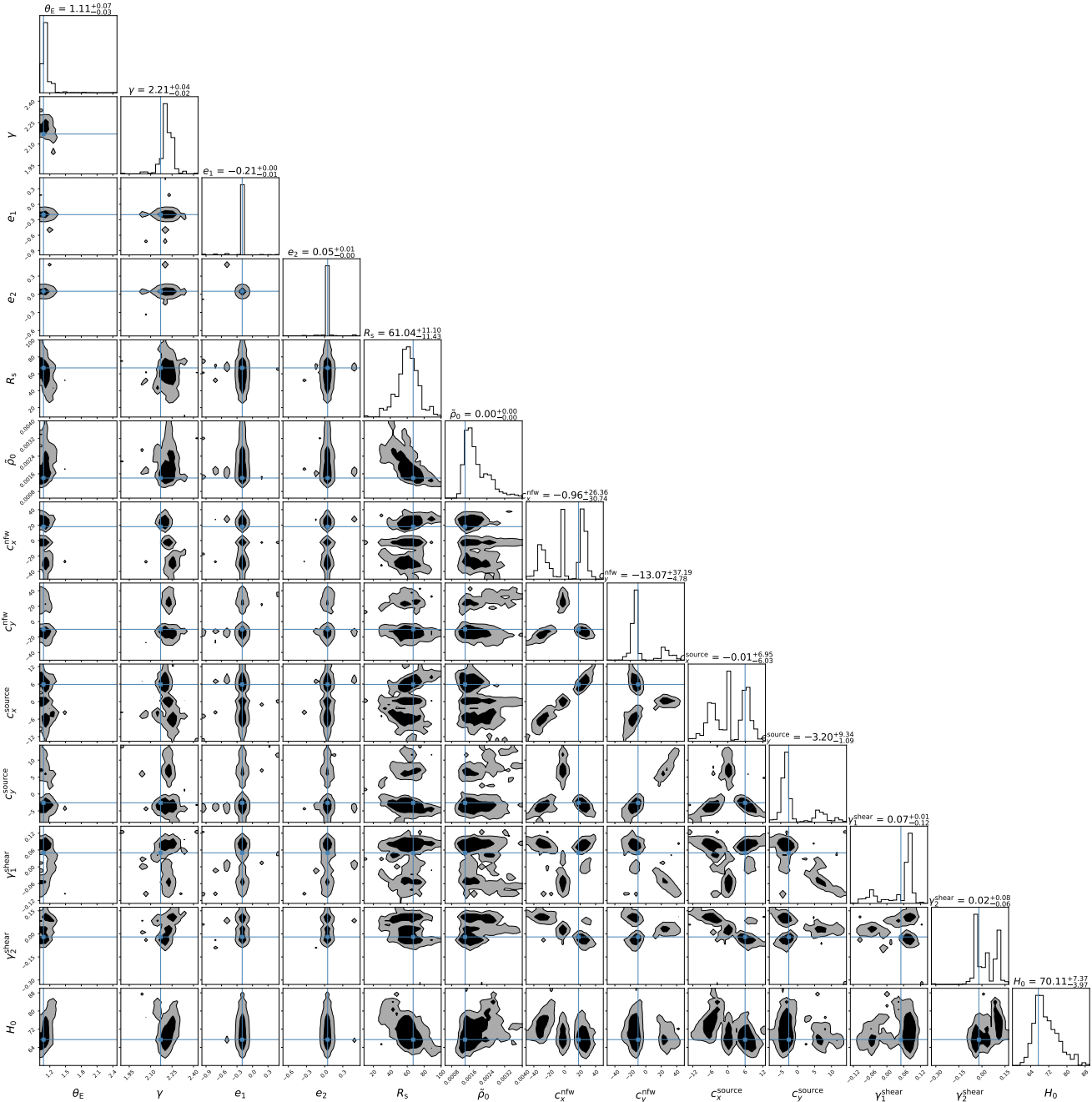
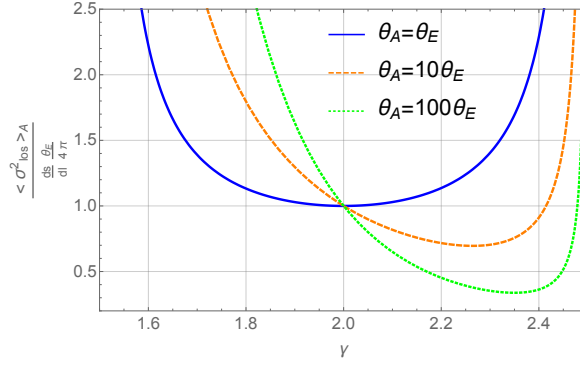
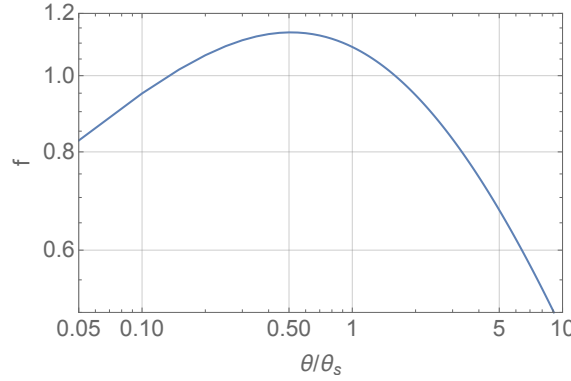


FIGURE 4.12: Mock analysis: zeus run, comprehensive view $\odot$. A cautionary comment is needed: the run has not converged. The main point of this run is to qualitatively show the three minima structure. No quantitative results are obtained from this run.

FIGURE 4.13: γ - and θ_A -dependence of aperture-averaged LOSVD.FIGURE 4.14: The LOSVD θ -dependence for the NFW profile.

To explore one of those effects, consider varying γ in the relation between σ_{los}^2 and θ_E . For non-SIS systems, $\sigma_{\text{los}}^2(\theta)$ depends on θ , and a relevant observable is the brightness-weighted average of σ_{los}^2 in some aperture θ_A . We can compare such averaged LOSVD to the SIS relation:

$$\frac{\langle \sigma_{\text{los}}^2 \rangle_A}{\frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi}} = \frac{1}{\frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi}} \frac{\int_0^{\theta_A} d\theta s_*(\theta) \sigma_{\text{los}}^2(\theta)}{\int_0^{\theta_A} d\theta s_*(\theta)} = \frac{3-\gamma}{5-2\gamma} \frac{\sqrt{\pi} \Gamma^2\left(\frac{\gamma}{2}\right) \Gamma\left(\gamma - \frac{3}{2}\right)}{\Gamma^2\left(\frac{\gamma-1}{2}\right) \Gamma(\gamma)} \left(\frac{\theta_A}{\theta_E}\right)^{2-\gamma}. \quad (4.24)$$

We illustrate Eq. (4.23) in Fig. 4.13.

NFW

For the NFW profile, assuming again that the tracer galaxy density follows the mass density, we can write:

$$\sigma_{\text{los}}^2(\theta) = G\rho_0 R_s^2 f\left(\frac{\theta}{\theta_s}\right), \quad f(a) = \frac{8\pi a \int_1^\infty \frac{dy y}{\sqrt{y^2-1}} \int_{ya}^\infty \frac{dx}{x^3(1+x)^2} \left(\ln(1+x) - \frac{x}{1+x}\right)}{2 \int_0^\infty \frac{dz}{\sqrt{a^2+z^2}(1+\sqrt{a^2+z^2})^2}}. \quad (4.25)$$

Some of these integrals can be done in closed form, but that is not particularly illuminating. The function $f(\theta/\theta_s)$ is shown in Fig. 4.14.

Assuming the NFW model, a kinematics prior should use Eq. (4.25) (aperture-averaged as needed) to translate a LOSVD measurement into a constraint on the combination of ρ_0 and R_s appearing in

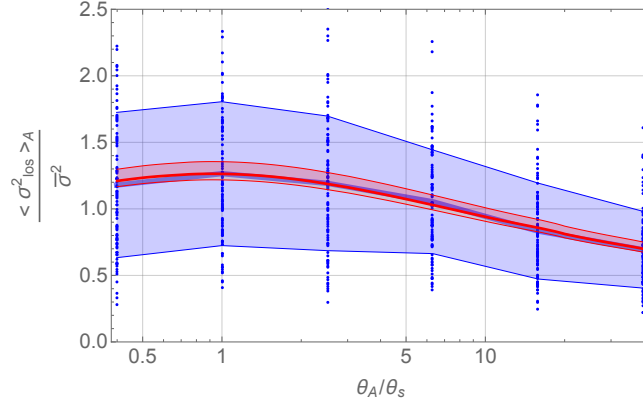


FIGURE 4.15: Aperture-averaged LOSVD for NFW, compared to the quantity $\bar{\sigma}^2 = GM_{\text{vir}}/(3R_{\text{vir}})$. Red line: using Eq. (4.25) in Eq. (4.24), replacing $\frac{D_s}{D_{\text{ls}}} \frac{\theta_E}{4\pi}$ by $\bar{\sigma}^2$, with $c_{\text{vir}} = 3.5$. Red shaded band: varying $c_{\text{vir}} \rightarrow 3.5 \times 10^{\pm 0.14}$. Dots: direct calculation of σ_{los}^2 from mock groups, assuming 13 tracer galaxies per realization (see Sec. 4.2.6). Blue line (somewhat hidden under the red) and blue shaded band: mean and 1σ region for the mocks.

the equation. However, analyses in the literature took a somewhat different route. Ref. [Che+19] (after [KG00; Won+11]) considered the following expression as a proxy for the LOSVD of the NFW profile:

$$\bar{\sigma}^2 = \frac{GM_{\text{vir}}}{3R_{\text{vir}}}, \quad (4.26)$$

where $R_{\text{vir}} = c_{\text{vir}} R_s$, $M_{\text{vir}} = M(R_{\text{vir}})$, and c_{vir} is the NFW concentration parameter, for which it is possible to extract a theoretical prediction from N-body simulations [MDB08]. Namely, instead of using the physical relation, Eq. (4.25), to convert the measured LOSVD into a constraint on the group halo model, Ref. [Che+19] made the identification $\sigma_{\text{los}}^2 \rightarrow \bar{\sigma}^2$, and then used Eq. (4.26) to constrain M_{vir} and R_{vir} . The dependence of the RHS of Eq. (4.26) on c_{vir} can be clarified by noting that

$$\bar{\sigma}^2 = G\rho_0 R_s^2 \frac{4\pi}{3c_{\text{vir}}} \left(\ln(1 + c_{\text{vir}}) - \frac{c_{\text{vir}}}{1 + c_{\text{vir}}} \right). \quad (4.27)$$

The dependence is not strong: the factor $\frac{4\pi}{3c_{\text{vir}}} \left(\ln(1 + c_{\text{vir}}) - \frac{c_{\text{vir}}}{1 + c_{\text{vir}}} \right)$ is equal to $\{0.89, 0.85, 0.80, 0.62\}$ when varying $c_{\text{vir}} = \{3, 4, 5, 10\}$, respectively. However, it is noteworthy that this dependence is not physical, but rather introduced artificially. The red solid line in Fig. 4.15 compares the physical aperture-weighted LOSVD to the quantity $\bar{\sigma}^2$, using $c_{\text{vir}} = 3.5$, relevant for low-redshift massive galaxies.

The meaning of Fig. 4.15 is that using Eq. (4.26) and identifying $\sigma_{\text{los}}^2 \rightarrow \bar{\sigma}^2$ can introduce a systematic error. The magnitude of the error depends on the aperture and on the value adopted for c_{vir} , but can amount to $\sim 20\%$.

Kinematical and cosmological priors

Here we describe our attempt to define a kinematics+cosmology prior, following a similar procedure as used in [Che+19] for PG1115+080.

The first step taken by [Che+19] is to identify $\sigma_{\text{los}} \rightarrow \bar{\sigma}$.

The next step is to invoke theoretical expectations based on N-body simulations. Following App. A

of [Won+11], the virial mass and the virial radius of the halo are related to a characteristic redshift-dependent overdensity:

$$M_{\text{vir}} = \frac{4\pi}{3} R_{\text{vir}}^3 \Delta_c(z) \rho_c(z), \quad (4.28)$$

where $\Delta_c(z) = 178(\rho_m(z)/\rho_c(z))^{0.45}$ is the expected halo overdensity at the virial radius and $\rho_c(z)$ and $\rho_m(z)$ are the cosmological critical density and matter density at redshift z [ENS01]. Combining Eqs. (4.28) and (4.26) turns $\bar{\sigma}$ into separate priors on R_{vir} and M_{vir} :

$$R_{\text{vir}} = \frac{3\bar{\sigma}}{2\sqrt{\pi G \Delta_c \rho_c}}, \quad M_{\text{vir}} = \frac{9\bar{\sigma}^3}{2\sqrt{\pi G^3 \Delta_c \rho_c}}. \quad (4.29)$$

Next, Ref. [Che+19] adds a prior on c_{vir} , using the $M_{\text{vir}} - c_{\text{vir}}$ relation from Ref. [MDB08]:

$$\log_{10} c_{\text{vir}} \approx 0.97 - 0.09 \log_{10} \left(\frac{M_{\text{vir}} h}{10^{12} M_{\odot}} \right). \quad (4.30)$$

Treating c_{vir} as a function of $\bar{\sigma}$ via Eqs. (4.30) and (4.29) right, one has now produced separate priors on the parameters ρ_s and R_s :

$$\rho_s = \frac{\Delta_c \rho_c c_{\text{vir}}^3}{3} \left(\ln(1 + c_{\text{vir}}) - \frac{c_{\text{vir}}}{1 + c_{\text{vir}}} \right)^{-1}, \quad R_s = \frac{R_{\text{vir}}}{c_{\text{vir}}}. \quad (4.31)$$

The uncertainty can be estimated by combining the observational uncertainty on σ_{los} and the theoretical scatter on the $M_{\text{vir}} - c_{\text{vir}}$ relation; the latter is taken in Ref. [Che+19] as ± 0.14 on the $\log_{10} c_{\text{vir}}$ expression in Eq. (4.30).

In practice, our implementation of these priors in the MCMC mock analysis is as follows:

1. The parameters that are directly sampled by the chain, defining the posterior likelihood space, are angular variables that are linearly proportional to ρ_s and R_s . In each step of the calculation, we translate the sampled value of ρ_s into an expected value of c_{vir} , using Eq. (4.31).
2. Having obtained c_{vir} , we convert Eq. (4.30) into an equation for R_{vir} , replacing M_{vir} by R_{vir} using Eq. (4.28). Now, we implement the estimated theoretical scatter of ± 0.14 dex on c_{vir} in the $M_{\text{vir}} - c_{\text{vir}}$ relation, to define a range of acceptable values for R_s via $R_s \in R_{\text{vir}} / (10^{+0.14}, 10^{-0.14})$. In the chain, we discard sampled values of R_s that fall outside of this range. This step therefore enforces a redshift-dependent correlation between ρ_s and R_s . Up to this point, we made no connection to kinematics.
3. Finally we come to the kinematics. Using Eq. (4.27), we convert the model point represented by ρ_s and R_s , along with the central value of c_{vir} , into a model prediction for $\bar{\sigma}$. This prediction is then compared to the measured value of σ_{los} using the nominal observational uncertainty to obtain a likelihood factor.

4.2.6 Sample variance

Sample variance is the dominant nominal source of uncertainty in σ_{los}^2 , given that typically only a handful of galaxies are measured as tracers of the group. Ref. [Wil+17] used bootstrap to estimate sample variance directly from the measured sample of galaxies; here, we complement this route by generating random sets of N_g galaxies tracing an NFW halo.

The realizations are drawn from an equilibrium phase space distribution function $f(r, v)$. For a spherical halo with a statistically static distribution function, we have $f(r, v) = f(\mathcal{E}(r, v))$, see Eq. (5.25). We calculate $f(\mathcal{E})$ numerically, and use it to draw samples of N_g tracer galaxies that fall within projected

aperture θ_A . The results of this exercise for $N_g = 13$ (as in PG1115+080) and for different apertures are shown as blue dots in Fig. 4.15. For each value of θ_A/θ_s we generate 100 mock samples. For each sample we calculate σ_{los}^2 directly as the variance of LOS velocity across the N_g galaxies. The mean and standard deviation of σ_{los}^2 are shown by the thick blue line and shaded region.

The variance we find for σ_{los}^2 in Fig. 4.15 is roughly consistent with sample variance of a normal distribution: $\frac{\delta\sigma_{\text{los}}^2}{\sigma_{\text{los}}^2} \approx \sqrt{\frac{2}{N_g-1}} \approx 0.4$ for $N_g = 13$. For comparison with Ref. [Wil+16], we also calculate uncertainty estimates for σ_{los}^2 using the bootstrap method. As a rule, the bootstrap method provides a slightly lower uncertainty estimate than the variance found with mock realizations, but the difference is small: whereas direct sample variance predicts $\Delta\sigma_{\text{los}}^2/\sigma_{\text{los}}^2 \approx 0.4$, bootstrap predicts $\Delta\sigma_{\text{los}}^2/\sigma_{\text{los}}^2 \approx 0.35$. Thus, we reproduce the sample variance-dominated uncertainty estimate of Ref. [Wil+17] for PG1115+080.

4.2.7 Estimating the probability of an MCMC to fall into a displaced minimum

As noted in Sec. 4.2.3, at least for a wide group centroid prior (as derived in Ref. [Wil+17] for PG1115+080), trial and error with emcee [FM+13] suggests that the initial placement of the walkers is a key factor in deciding which likelihood minimum will attract the fit. We can (roughly) estimate the probability of falling into a wrong minimum by the probability for the walker placement to start off closer to a false minimum than to the truth one. Let us assume that the initial placement of the walkers is chosen to coincide with the prior's center (this seems like a natural choice). In this case, the probability to fall into a wrong minimum is roughly given by the probability of the group centroid prior to be nearer a false minimum than the truth one.

We can estimate this probability using mock samples of tracer galaxies as in Sec. 4.2.6. Consider a sample of N_g galaxies, and choose a random member to be the “primary lens”. From the same sample, derive a group center prior as the center of mass of the members. In general, of course, the prior center does not coincide with the center of the halo used to generate the mock.⁷ Now, draw a line connecting the primary lens with the true halo center, and another line connecting the primary lens with the prior center: the prior center is closer to a false minimum if the smaller angle between these lines is larger than 60° .

The result of this calculation depends on the number of member galaxies N_g , the group halo profile, and the analysis aperture. For $N_g = 13$ in the NFW model with an aperture of $10 R_s$ (around 3 virial radii), we find the wrong minimum probability to be $\sim 10\%$.

It should be clear that the simple estimate we described here ignores a range of possible selection effects, both natural (such as mass segregation, and the bias of more massive group members to become primary lenses) as well as analysis-specific (such as luminosity-dependent contamination and incompleteness). Such effects can probably modify our estimates at the $\mathcal{O}(1)$ level, but should not change the order of magnitude of the result.

4.2.8 How common is the flexion degeneracy?

PG1115+080 is proof of concept that lensing set-ups like the one we analyzed are observationally relevant. But since PG1115+080 was selected specifically because of its massive, near-by group association, this set-up may be quite uncommon. Most of the lensed quasars analyzed for time-delay cosmography may either not be associated with LOS groups; or, if they are, may lie far from the group's projected center of mass, making flexion terms less important than for PG1115+080. A detailed analysis of the fraction of systems that may exhibit flexion degeneracy is beyond the scope of this work. Nevertheless we can make a crude estimate using the data from [Wil+17], as follows.

⁷The statistical distribution of the mismatch, which goes to defining the prior width, can be estimated either by bootstrap, as done in [Wil+17] for the actual data, or by repeated mocks. We test both, and find them to be compatible.

Tab. 1 in Ref. [Wil+17] lists reconstructed properties of LOS groups (including primary lens host where exists) of their 26 lens systems. For each LOS group, estimated values of M_{vir} and R_{vir} are quoted, along with the measured LOSVD, denoted there by σ_{grp} . We insert M_{vir} into Eq. (4.30) to estimate c_{vir} ; then use σ_{grp} , R_{vir} , and c_{vir} in Eq. (4.27) to estimate ρ_0 . Using the quoted group redshift to calculate the relevant value of Σ_{cr} , Eq (D.46) allow us to obtain $\Delta\kappa$. For each system, we estimate the rough magnitude of the flexion by the root-mean-square $\sqrt{\sum_i (\Delta\kappa_i/h_i)^2}$, including all LOS groups i , and estimate the corresponding deflection angle near the primary lens by $\Delta\theta \sim \sqrt{\sum_i (\Delta\kappa_i/h_i)^2} \theta_E^2$, taking $\theta_E = 1''$ for definiteness.

Of the 26 systems considered in [Wil+17], PG1115+080 does indeed come up with the highest flexion estimate of $\Delta\theta \sim 0.02''$. The next highest system is HE0435, with $\Delta\theta \sim 0.005''$. Altogether, 3 systems (7 systems) out of the 26 have $\Delta\theta \gtrsim 0.0025''$ ($\Delta\theta \gtrsim 1$ mas).

Numerical experiments with mock analyses mimicking the main features of the pipeline of [Mil+20] suggest that flexion terms are quantitatively important in the fit if the flexion-induced deflection angle is larger than $\sim 0.001''$. Very roughly, we can expect that flexion degeneracy should be a concern in the same parametric regime. It is interesting to note that all of the 4 systems that take part in both of the cosmography [Mil+20] and kinematics [Wil+17] campaigns, turn out to exhibit $\Delta\theta \gtrsim 0.001''$.

4.2.9 Summary of the flexion host-group discrete degeneracy

In this section (Sec. 4.2), which contains the main findings of Ref. [TB23], we showed an approximate threefold discrete modelling degeneracy, associated with rotating the assumed position of the group centroid by an angle of $2\pi/3$ around the primary lens. This produces a posterior likelihood with three disjoint minima. MCMC algorithms that fail to expose this structure may fall into a false minimum. The inferred value of H_0 found in a false minimum is systematically biased high. Using numerical mock data experiments motivated by the PG1115+080 system, we demonstrated that the H_0 bias can reach $\sim 10\%$. Bayesian cosmography analyses should explore the full posterior likelihood landscape. Awareness of this possible three disjoint minima structure, if not already there, is needed, to be able to reach a precise and accurate H_0 determination from time delays.

The choice of the minimum detected by the MCMC can strongly depend on the starting position of the walkers in the space of group centroid coordinates. If the starting point is chosen as the centroid prior center, then the probability for the MCMC to land in a displaced minimum may be rather small. For a sample of 13 tracer galaxies (relevant for PG1115+080) with an aperture of about 3 virial radii in an NFW halo, this probability is $\sim 10\%$.

Our analysis suggests the following recommendations.

1. Bayesian cosmography analyses should explore the full posterior likelihood landscape. Awareness of this possible three disjoint minima structure, if not already there, is needed. As an example, the zeus algorithm can expose the full structure, while emcee tends to detect only one local minimum. The choice of the minimum seems to be dictated by the starting position of the walkers in the algorithm and/or the presence of Gaussian priors on the coordinate centers of the halo, c_i^{nfw} (a displaced Gaussian prior with respect to the truth will tend to fall into a minimum which does not contain the true c_i^{nfw}).
2. Tracer galaxy kinematics and group centroids are the only handles in constraining the MSD associated with the group, hence it is important to use them to constrain the parameters of the halo model in the correct way. We pinpoint that, given the simplistic assumption Eq. (4.26) which relates the kinematics measurement with halo parameters, one should take into account the scatter in this relation, effectively increasing the uncertainty on σ_{los} .

3. X-ray data may help to pinpoint the centroid of a massive group, resolving the degeneracy.
4. As an aside, we note that some cosmography analyses (e.g. [Che+19]) used an incorrect kinematics prior to constrain the group model. The error in the interpretation of tracer galaxy velocity dispersion depends on the analysis aperture, and can reach $\sim 20\%$. A correct version of the kinematics prior is reviewed in Sec. 4.2.5.

The number of strong lensing time delay systems is expected to increase by more than an order of magnitude in the near future [OM10; Lia+15; Dob+09], an important step towards possibly reaching a few percent lensing determination of H_0 [BT21a]. This program could be further assisted by many resources [Tre+18; Bac+20; Bla+20; Abe+09; Gar+06]. Our study highlights some pitfalls (and suggests solutions) that need to be taken into account if the precision goal for H_0 should also be accurate.

4.3 Large scale structures and multiple source systems

The prospects for time delay cosmography will greatly benefit from various upcoming surveys [Tre+18; Bac+20; Bla+20] and with the JWST [Gar+06], that will sharpen constraints on lens stellar kinematics [BT21a; Y1+21]. Notably, LSST is expected to give us a sample of $\mathcal{O}(100)$ double-source-plane galaxy-scale lenses [Man+18], among $\mathcal{O}(10^4)$ lensed quasars and $\mathcal{O}(100)$ lensed supernovae Ia on conservative estimates [OM10; Lia+15; Dob+09]. Multiple source systems are particularly interesting, since they can give us more information on galaxy lens profiles, and possibly they can allow us to mitigate the effect of degeneracies like the MSD. The prospects coming from these new data call for proper assessing of degeneracies in strong gravitational lensing.

In [Teo+22], we discussed the effects of LSS on the line of sight contribution to the H_0 measurement, together with a discussion of the advantage that multiple source systems might have in fighting the degeneracies entering the measurement of H_0 . In this section, we report the main results of Ref. [Teo+22]. We describe the multiple source MSD in Sec. 4.3.1; we investigate the possibility to constrain the external convergence via stellar kinematics in Sec. 4.3.2; we point out what is really observable in multiple source systems, connected to Einstein angle ratios between multiple sources, in Sec. 4.3.3 and Sec. 4.3.4; we give estimates of the external convergence factors entering such observables in Sec. 4.3.5; we apply our findings to some real systems in Sec. 4.3.6 and summarize in Sec. 4.3.7.

4.3.1 The multiple source MSD

We will take the multiple lens equation Eq. (2.67), but we will neglect the multiple deflectors influence $\vec{\alpha}_i$ for $i > 0$ to focus on LSS effects. Deflectors influence add complexity to the model, and does not change the main conclusions discussed in the following. Hence, we take Eq. (2.67) in the form [Mir91; KS93; BK96; KKS97; McC+14; Sch19; FLU21]

$$\vec{\beta}_i = (1 - \kappa_i^s)(\mathbb{I} + \Gamma_i^s)\vec{\theta} - (1 - \kappa_i^{ls})(\mathbb{I} + \Gamma_i^{ls})C_{i0}\vec{\alpha}_0((1 - \kappa^l)(\mathbb{I} + \Gamma^l)\vec{\theta}), \quad (4.32)$$

where we used several shorthands, $\vec{\beta}_i = \vec{\beta}(w_i, 0)$, κ_i^r are external convergence factors for source i , $\kappa_i^s = \kappa(w_i, 0)$, $\kappa_i^{ls} = \kappa(w_i, w_1)$ and

$$\Gamma_i^r = - \begin{pmatrix} \gamma_1^{ri} & \gamma_2^{ri} \\ \gamma_2^{ri} & -\gamma_1^{ri} \end{pmatrix}, \quad (4.33)$$

is the reduced shear matrix, and the superscript $r = l, s, ls$ indicates observer-lens, observer-source, and lens-source lines of sight, the source labelled by i . Time delays between image A and B become

$$\Delta t_{AB}^i = D_{\text{dt}}^i \Delta \tau_{AB}^i, \quad (4.34)$$

$$\Delta \tau_{AB}^i = \frac{1}{2} \vec{\theta}_A^T \left(\mathbb{I} - M_i^s - M^l + M_i^{ls} \right) \vec{\theta}_A - \vec{\beta}^T \left(\mathbb{I} - M^l + M_i^{ls} \right) \vec{\theta}_A - \psi_i((\mathbb{I} - M^l) \vec{\theta}_A) - \{A \leftrightarrow B\}. \quad (4.35)$$

Compared with the internal convergences κ_i , which have $\kappa_i(\vec{\theta}_{E,i}) = \mathcal{O}(1)$, the weak lensing terms are $\mathcal{O}(0.1), \mathcal{O}(0.01)$.

In the generic multiple source case, with LSS effects, we can rewrite a more generic MSD transformation, by allowing M_i^{ls} and M^l to also be adjusted, alongside an adjustment of the intrinsic lens model and the modelled source coordinates:

$$M_i^s \mapsto M_i^{s,\lambda} = \lambda_s M_i^s + (1 - \lambda_s) \mathbb{I}, \quad (4.36)$$

$$M_i^{ls} \mapsto M_i^{ls,\lambda} = \lambda_{ls} M_i^{ls} + (1 - \lambda_{ls}) \mathbb{I}, \quad (4.37)$$

$$M^l \mapsto M^{l,\lambda} = \lambda_l M^l + (1 - \lambda_l) \mathbb{I}, \quad (4.38)$$

$$\vec{\beta}_i \mapsto \vec{\beta}_i^\lambda = \lambda_s \vec{\beta}_i, \quad (4.39)$$

$$\vec{\alpha}_0(\vec{\theta}) \mapsto \vec{\alpha}_0^\lambda(\vec{\theta}) = \lambda_s \lambda_{ls}^{-1} \vec{\alpha}_0(\lambda_l^{-1} \vec{\theta}), \quad (4.40)$$

$$\psi_0(\vec{\theta}) \mapsto \psi_0^\lambda(\vec{\theta}) = \lambda_s \lambda_{ls}^{-1} \lambda_l \psi_0(\lambda_l^{-1} \vec{\theta}). \quad (4.41)$$

Here λ_s, λ_{ls} , and λ_l are independent parameters. Note that Eq. (4.36) (for example) amounts to $\kappa_i^s \mapsto \kappa_i^{s,\lambda} = \lambda_s \kappa_i^s + (1 - \lambda_s)$, $\Gamma_i^s \mapsto \Gamma_i^{s,\lambda} = \lambda_s \Gamma_i^s$.

Inserting Eqs. (4.39-4.41) into Eq. (4.34), we see that the dimensionless time delay $\Delta \tau_{AB}$ of the transformed model changes according to:

$$\Delta \tau_{AB} \mapsto \Delta \tau_{AB}^\lambda = \lambda_s \lambda_{ls}^{-1} \lambda_l \Delta \tau_{AB}, \quad (4.42)$$

which implies the H_0 bias

$$H_0 \mapsto H_0^\lambda = \lambda_s \lambda_{ls}^{-1} \lambda_l H_0. \quad (4.43)$$

It is clear that the availability of multiple sources does not break the MSD, due to the modelling degeneracy of Eqs. (4.36-4.41) (similar conclusions, with a different version of the MSD with intermediate sources acting as additional strong lenses for background sources, can be found in Refs. [McC+14; Sch19].).

External convergence in single source. In an imaging analysis, if only a single source is available (say $i = 1$), one can use Eqs. (4.36-4.38) with the choice

$$\lambda_s = \frac{1}{1 - \kappa^s}, \quad \lambda_{ls} = \frac{1}{1 - \kappa^{ls}}, \quad \lambda_l = \frac{1}{1 - \kappa^l}, \quad (4.44)$$

to eliminate all of κ^s, κ^{ls} , and κ^l from the modelling. For this reason, the task of extracting lensing information in imaging data is often performed ignoring external convergence [Rus+20; Sha+20; Che+19; Won+20; Mil+20; Bir+19].

Suppose we denote the fit result for the “intrinsic deflection angle” in such an analysis by $\vec{\alpha}^{\text{model}}(\vec{\theta})$. By “eliminating external convergence from the equations”, we mean that the fit looks for a deflection angle model $\vec{\alpha}^{\text{model}}(\vec{\theta})$ which goes to zero at large $|\vec{\theta}|$, possibly up to a uniform shear term $\Gamma^s \vec{\theta}$. Eq. (4.40)

implies that $\vec{\alpha}^{\text{model}}(\vec{\theta})$ is related to the true underlying physical intrinsic deflection angle by

$$\vec{\alpha}^{\text{model}}(\vec{\theta}) = \frac{1 - \kappa^{\text{ls}}}{1 - \kappa^{\text{s}}} \vec{\alpha}((1 - \kappa^{\text{l}})\vec{\theta}), \quad (4.45)$$

where $\kappa^{\text{s}, \text{ls}, \text{l}}$ are the true physical values of the weak lensing terms. Given a measurement of the physical image time delays, and deriving the dimensionless time delay $\Delta\tau_{AB}^{\text{model}}$ from $\vec{\alpha}^{\text{model}}$, one can extract an inferred result H_0^{model} , which is related to the truth value H_0 by [Bir+20]

$$H_0^{\text{model}} = \frac{1 - \kappa^{\text{ls}}}{(1 - \kappa^{\text{s}})(1 - \kappa^{\text{l}})} H_0. \quad (4.46)$$

The usual challenge of the weak lensing MSD for cosmography is to constrain the correction factor $(1 - \kappa^{\text{ls}})/[(1 - \kappa^{\text{s}})(1 - \kappa^{\text{l}})] \approx 1 + \kappa^{\text{s}} + \kappa^{\text{l}} - \kappa^{\text{ls}}$.

Stellar kinematics is sensitive to the intrinsic mass-per-radius ($M(R)/R$) of the lens, and can be used to partially resolve the MSD. Refs. [Rus+20; Bir+19; Sha+20; Che+19; Won+20; Mil+20] used kinematics to constrain the MSD, but in these works, weak lensing was only parameterised in terms of κ^{s} , omitting κ^{ls} and κ^{l} . The omission of κ^{ls} and κ^{l} biases the inferred value of H_0 .

Ref. [Mil+20] presented a treatment of systematics in recent cosmographic analyses. There, the following expression was used to correct for the weak lensing MSD⁸:

$$H_0^{\text{inferred}} = (1 - \kappa^{\text{ext}}) H_0^{\text{model}}. \quad (4.47)$$

The terms $\kappa^{\text{ls}, \text{l}}$ were effectively set to zero in the modelling, as they were ignored in both kinematics and imaging. From Eq. (8) in Ref. [Mil+20], as we will show in next section,

$$1 - \kappa^{\text{ext}} := \frac{1 - \kappa^{\text{s}}}{1 - \kappa^{\text{ls}}}. \quad (4.48)$$

This expression coincides with the discussion in Ref. [BAR16], cited by [Mil+20] for the treatment of kinematics, if we set $\kappa^{\text{l}} = \kappa^{\text{ls}} \rightarrow 0$, in which case $\kappa^{\text{ext}} \rightarrow \kappa^{\text{s}}$.

4.3.2 Stellar kinematics and LSS

In this section we show how stellar kinematics is sensitive to κ^{ext} , Eq. (4.48). The stellar kinematics velocity dispersion is expected to obey the spherical Jeans equation Eq. (1.83). It is possible to reduce the Jeans equation to angle variables only; assuming $r = D_{\text{l,eff}}\theta$, where θ is the angle in the sky an observer on Earth sees, we have

$$\frac{1}{n_*} \frac{d(n_* \sigma_r^2(D_{\text{l,eff}}\theta))}{d\theta} + 2\beta_{\text{ani}}(D_{\text{l,eff}}\theta) \frac{\sigma_r^2(D_{\text{l,eff}}\theta)}{\theta} = -G \frac{M(D_{\text{l,eff}}\theta)}{D_{\text{l,eff}}\theta^2}; \quad (4.49)$$

Now, we have to distinguish two cases:

- Observer aware of weak lensing effects (truth, case 1): he will take correctly into account weak lensing effects and write $D_{\text{l,eff}} = D_1(1 - \kappa^{\text{l}})$, moreover he will use the correct expression for $M(R)$, obtained by the correct lens equation

$$\vec{\beta} = (1 - \kappa^{\text{s}})\vec{\theta} - (1 - \kappa^{\text{ls}})\vec{\alpha}((1 - \kappa^{\text{l}})\vec{\theta}). \quad (4.50)$$

⁸See discussion around Eqs.(7-8) and Eq. (16) in [Mil+20].

- Observer not aware of weak lensing effects (case 2): he will write $D_{l,\text{eff}} = D_l$, and he will attempt to recover the mass with the wrong model (wrong model quantities are expressed with a tilde)

$$\vec{\beta} = \vec{\theta} - \vec{\alpha}(\vec{\theta}), \quad \vec{\alpha}(\vec{\theta}) = \frac{1 - \kappa^{\text{ls}}}{1 - \kappa^{\text{s}}} \vec{\alpha}((1 - \kappa^{\text{l}})\vec{\theta}), \quad (4.51)$$

resulting with a mass inference $\tilde{M} \neq M$.

To understand the connection between $\tilde{M}$ and M , we can write, exploiting Eq. (2.36),

$$\begin{aligned} \vec{\alpha}(\vec{\theta}) &= \frac{1 - \kappa^{\text{ls}}}{1 - \kappa^{\text{s}}} \frac{1}{\pi} \int d^2\theta' \underbrace{\frac{(1 - \kappa^{\text{l}})\vec{\theta} - \vec{\theta}'}{|(1 - \kappa^{\text{l}})\vec{\theta} - \vec{\theta}'|^2} \kappa(\vec{\theta}')}_{=\vec{\alpha}((1 - \kappa^{\text{l}})\vec{\theta})} \\ &= \frac{1}{\pi} \int d^2\theta'' \frac{\vec{\theta} - \vec{\theta}''}{|\vec{\theta} - \vec{\theta}''|^2} \underbrace{\frac{(1 - \kappa^{\text{ls}})(1 - \kappa^{\text{l}})}{1 - \kappa^{\text{s}}} \kappa((1 - \kappa^{\text{l}})\vec{\theta}'')}_{=\tilde{\kappa}(\vec{\theta}'')} \end{aligned} \quad (4.52)$$

where we changed variables in the second step, $\vec{\theta}' = (1 - \kappa^{\text{l}})\vec{\theta}''$. This implies

$$\begin{aligned} \kappa((1 - \kappa^{\text{l}})\vec{\theta}) &= \frac{\int dr_z \rho(r_z, D_l(1 - \kappa^{\text{l}})\vec{\theta})}{\Sigma_{\text{cr}}} = D_l(1 - \kappa^{\text{l}}) \frac{\int d\theta_z \rho(\theta_z, D_l(1 - \kappa^{\text{l}})\vec{\theta})}{\Sigma_{\text{cr}}} \\ &= \frac{1 - \kappa^{\text{s}}}{(1 - \kappa^{\text{ls}})(1 - \kappa^{\text{l}})} \tilde{\kappa}(\vec{\theta}) = \frac{1 - \kappa^{\text{s}}}{(1 - \kappa^{\text{ls}})(1 - \kappa^{\text{l}})} D_l \frac{\int d\theta_z \tilde{\rho}(\theta_z, \vec{\theta})}{\Sigma_{\text{cr}}}, \end{aligned} \quad (4.53)$$

which implies

$$\rho(\theta_z, D_l(1 - \kappa^{\text{l}})\vec{\theta}) = \frac{1 - \kappa^{\text{s}}}{(1 - \kappa^{\text{ls}})(1 - \kappa^{\text{l}})^2} \tilde{\rho}(\theta_z, \vec{\theta}). \quad (4.54)$$

This straightforwardly translates into

$$M(D_l(1 - \kappa^{\text{l}})\theta) = (D_l(1 - \kappa^{\text{l}}))^3 \int d\Omega_2 \int_0^\theta \theta'^2 d\theta' \rho(\theta_z, D_l(1 - \kappa^{\text{l}})\vec{\theta}) = \frac{(1 - \kappa^{\text{s}})(1 - \kappa^{\text{l}})}{1 - \kappa^{\text{ls}}} \tilde{M}(D_l\vec{\theta}). \quad (4.55)$$

The two Jeans equation observer one and two will write respectively are

$$(1) \quad \frac{1}{\rho_*} \frac{d(\rho_* \sigma_r^2(D_l(1 - \kappa^{\text{l}})\theta))}{d\theta} + 2\beta_{\text{ani}}(D_l(1 - \kappa^{\text{l}})\theta) \frac{\sigma_r^2(D_l(1 - \kappa^{\text{l}})\theta)}{\theta} = -G \frac{M(D_l(1 - \kappa^{\text{l}})\theta)}{D_l(1 - \kappa^{\text{l}})\theta^2}, \quad (4.56)$$

$$(2) \quad \frac{1}{\rho_*} \frac{d(\rho_* \tilde{\sigma}_r^2(D_l\theta))}{d\theta} + 2\beta_{\text{ani}}(D_l\theta) \frac{\tilde{\sigma}_r^2(D_l\theta)}{\theta} = -G \frac{\tilde{M}(D_l\theta)}{D_l\theta^2}, \quad (4.57)$$

one can see that

$$\sigma_r^2((1 - \kappa^{\text{l}})D_l\theta) = \frac{1 - \kappa^{\text{s}}}{1 - \kappa^{\text{ls}}} \tilde{\sigma}_r^2(D_l\theta); \quad (4.58)$$

in particular, the κ^{l} dependence drops. The same applies for σ_{los}^2 , Eq. (1.96), as can be trivially shown. Eq. (4.58) tells us that stellar kinematic is sensitive to κ_{ext} as defined in Eq. (4.48).

Combining Eqs. (4.48), (4.47), and (4.46), we conclude that in Ref. [Mil+20] the relation between the inferred value and the truth value of the Hubble parameter was biased by the following factor:

$$\frac{H_0^{\text{inferred}}}{H_0} = \frac{1 - \kappa^{\text{s}}}{1 - \kappa^{\text{ls}}} \frac{1 - \kappa^{\text{ls}}}{(1 - \kappa^{\text{s}})(1 - \kappa^{\text{l}})} \approx 1 + \kappa^{\text{l}}. \quad (4.59)$$

We should note that although we considered the LOSVD as an observable, in practice it is not directly measured. Various observational effects such as luminosity weighting, point spread function, etc., must be taken into account. Moreover, there are important theoretical uncertainties due to the velocity anisotropy, and also due to the actual lens halo density profile, which must be marginalized over in the likelihood.

4.3.3 Multiple sources and LSS

If multiple sources are available, then the MSD requires a simultaneous adjustment of the weak lensing variables for all sources. In particular, the $\kappa_i^{s,ls}$ parameters for all sources i must be adjusted together, following Eqs. (4.36,4.37). Therefore, in the multi-source scenario, a certain combination of external convergence terms is measurable from the imaging data. A quick way to see what this measurable combination is, is by assuming that the analysis pipeline attempts to fit the two systems i and j separately and independently, omitting external convergence from the equations using Eq. (4.45). The outcome of such a procedure are two independent fits for the effective deflection angle, the results of which should be related by an over-all factor:

$$\frac{|\vec{\alpha}_{j-1}^{\text{model}}(\vec{\theta})|}{|\vec{\alpha}_{i-1}^{\text{model}}(\vec{\theta})|} = \frac{C_{j0}}{C_{i0}} \frac{1 - \kappa_j^{ls}}{1 - \kappa_i^{ls}} \frac{1 - \kappa_i^s}{1 - \kappa_j^s} \approx \frac{C_{j0}}{C_{i0}} \left(1 + \delta\kappa_{ji}^s - \delta\kappa_{ji}^{ls} \right), \quad (4.60)$$

where $\delta\kappa_{ji}^r := \kappa_j^r - \kappa_i^r$. Assuming Einstein rings $\theta_{E,i}$ for the various sources can be distinguished, then one can substitute the previous with

$$\frac{\theta_{E,j}}{\theta_{E,i}} \approx \frac{C_{j0}}{C_{i0}} \left(1 + \delta\kappa_{ji}^s - \delta\kappa_{ji}^{ls} \right). \quad (4.61)$$

The left hand side of Eq. (4.60) is measurable, and the C_{i0} factors are known, so the combination $(\kappa_j^{ls} - \kappa_i^{ls}) - (\kappa_j^s - \kappa_i^s)$ is, in principle, measurable.

Unfortunately, as this combination of terms is invariant under the MSD, it cannot resolve the MSD impact on the H_0 inference.

4.3.4 Multiple sources and the core-MSD

Uncertainties in the intrinsic mass profile of the lens, discussed in Sec. 4.1, could pose a more serious problem to time-delay measurements of H_0 , than that posed by weak external convergence. Specifically, an extended cored density component in lens galaxies would act similarly to external convergence [SS13b; BCS20], but could, in principle, cause a much larger effect. This scenario could occur in the ULDM model, as we will discuss in Ch. 5.

The core term is mathematically identical to a redefinition of the observer-source external convergence. Considering Eq. (4.1), we see that at the level of the modeling of imaging data, the core component is indistinguishable from the shift

$$M_i^s \rightarrow M_i^s + (\mathbb{I} - M_i^{ls})(\mathbb{I} - M^l)C_{i0}\kappa_{c0}. \quad (4.62)$$

We will think of the internal core convergence κ_{c0} as a small parameter, albeit potentially somewhat larger than cosmological weak external convergence terms. We have in our mind $\kappa_{c0} \approx 0.1$, which would bias up H_0 measurements by 10%.

Expressed in terms of convergence and shear parameters, at leading order in weak lensing terms, we have

$$\kappa_i^s \rightarrow \kappa_i^s + C_{i0}\kappa_{c0}(1 - \kappa_i^{\text{ls}} - \kappa^{\text{l}}), \quad (4.63)$$

$$\Gamma_i^s \rightarrow \Gamma_i^s - C_{i0}\kappa_{c0}(\Gamma_i^{\text{ls}} + \Gamma^{\text{l}}). \quad (4.64)$$

With this understanding one can see that imaging data alone cannot directly separate a core component from weak lensing. One must resort to kinematics analyses, or to theoretical considerations that could limit the plausible weak lensing effect. We focus on the latter.

It is worthwhile to highlight a key difference between convergence and shear. Under Eqs. (4.36-4.38), which deal purely with the modeling of external weak lensing, shear is adjusted multiplicatively, while convergence receives an additive correction. This feature is modified in Eqs. (4.63-4.64), but a key part of it remains manifest: the addition of the core adjusts Γ^s via an additive term, however that additive term is itself proportional to the shear terms $\Gamma^{\text{ls}} + \Gamma^{\text{l}}$. As a result, even if κ_{c0} is somewhat larger than typical weak lensing effects (e.g. $\kappa_{c0} \approx 0.1$), this still only amounts to a relative correction of $\sim 10\%$ in Γ^s . Constraining such a small effect observationally or theoretically would be challenging. This point is important because certain combinations of weak lensing shear terms can, in principle, be measured directly from imaging data [Bir+17; FLU21]. For convergence, Eq. (4.63) suggests a potentially large additive readjustment of κ^s , if κ_{c0} is larger than typical weak lensing effects. However, if only one source is available ($i = 1$), then it could be difficult for imaging data alone to constrain κ_{c0} .

With a core-MSD, Eq. (4.60) can be rewritten, inserting Eq. (4.63) into Eq. (4.60), and neglecting the small correction factor $1 - \kappa_i^{\text{ls}} - \kappa^{\text{l}} \approx 1$ in Eq. (4.63), as:

$$\frac{C_{i0}}{C_{j0}} \left| \frac{\vec{a}_{j-1}^{\text{model}}}{\vec{a}_{i-1}^{\text{model}}} \right| \approx 1 + (\kappa_j^s - \kappa_i^s) - (\kappa_j^{\text{ls}} - \kappa_i^{\text{ls}}) + \kappa_{c0} (C_{j0} - C_{i0}). \quad (4.65)$$

At a first glance in Eq. (4.65), one could hope that multiple source systems could resolve the core-MSD ambiguity, because the last term on the right hand side contains the large additive term $\propto \kappa_{c0}$. However, a second glance reveals a setback: in Eq. (4.65), κ_{c0} appears multiplied by the factor $C_{j0} - C_{i0}$, which involves ratio of angular diameter distances (see Eq. (2.64)). Unfortunately, the angular diameter distance is a non-monotonous function of redshift; moreover, the sources of typical strong lensing systems are often located between $z \sim 1$ and $z \sim 2.5$, that is, around the shallow maximum of $D_A(z, 0)$. As a result, in many systems of interest, the difference $C_{j0} - C_{i0} \sim \mathcal{O}(0.1)$ is much smaller than unity. This “dilutes” the efficiency at which imaging data in multiple source systems could constrain the internal MSD.

Fig. 4.16 illustrates our point. We show two examples of the curve $C_{20} - 1$. The blue line is inspired by the multiple source system of the cluster lens MACS1149.5+2223 [Gri+18; Gri+20]. The primary lens (cluster) redshift is $z_1 \approx 0.5$. Time-delays are measured for a type Ia supernova (source 1) at $z_1 \approx 1.5$. Fig. 4.16 shows $C_2 - 1$ as function of a second source redshift z_2 . (Actual additional sources of this system are distributed between $z_2 \sim 1.2$ and $z_2 \sim 3.7$.) The orange line is inspired by the galaxy lensing system DES J0408-5354 [Sha+20], $z_1 \approx 0.6$, with time-delays measured to a quasar at $z_1 \approx 2.3$.

Because $C_{j0} - C_{i0}$ is a small number, the κ_{c0} term in Eq. (4.65) could be diluted down to the natural scale of weak cosmological convergence. To detect (or constrain) an internal core, it therefore becomes crucial to estimate the magnitude of weak differential convergence. We will use the method outlined in Sec. 1.1.6.

Before we move on, let us make a rough assessment of the precision by which the LHS of (4.65) can actually be measured. Note that most of the information in the lensing data comes from the angular range $\theta \sim \theta_E$, where for simplicity of this estimate we can consider spherically symmetric systems

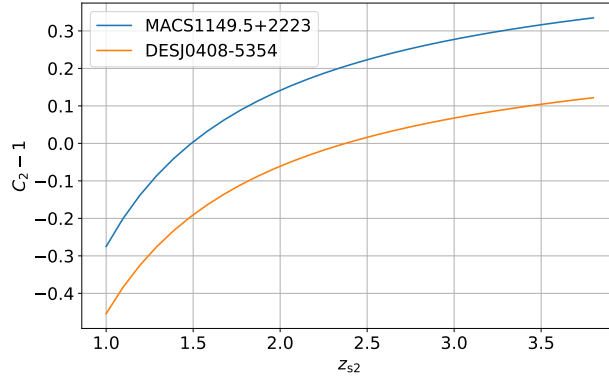


FIGURE 4.16: The angular diameter distance combination C_2 , appearing in lensing analyses with multiple sources. Code: $\mathfrak{Q}$.

and drop the vector notation on θ . Using the fact that $\alpha(\theta_E) = \theta_E$, the relative uncertainty by which $|\vec{\alpha}_{j-1}/\vec{\alpha}_{i-1}|$ can be measured is of similar size as the quoted precision on the ratio of Einstein angles, $|\theta_{Ej}/\theta_{Ei}|$. For typical TDCOSMO systems, this precision is at the level of $\sim 1\%$. Of course, this quoted precision corresponds to the main source considered by the analysis (usually, the source for which time-delays are measured). What we actually need is the differential convergence, and the precision on that would be dominated, given two sources $i = 1, 2$, by the source for which the precision on $\theta_{E,i}$ is poorest.

4.3.5 External convergence influence estimate

The typical angular separation of multiply-lensed sources in galaxy lensing campaigns is in the ballpark of arcseconds. This means that the proper transverse distance between the relevant geodesics is smaller than ~ 10 kpc, which is a small separation with respect to LSS. In the following, we will therefore compute the cosmological correlators at the same line of sight, $\hat{n} = \hat{n}'$. With this simplification, the integral for the variance of differential convergence, building on what we showed in Sec. 2.4, reads

$$\begin{aligned} \langle (\kappa(w_l, w_m; \hat{n}) - \kappa(w_n, w_o; \hat{n}))^2 \rangle &= \frac{9H_0^4 \Omega_{m,0}^2}{2(2\pi)^2} \int dw \int dw' \\ &\times q_{lmno}(w, w') \int dk k^2 j_0(k(w - w')) P_\delta(k, w, w'), \end{aligned} \quad (4.66)$$

with

$$q_{lmno}(w, w') := q_{lm}(w)q_{lm}(w') + q_{no}(w)q_{no}(w') - 2q_{lm}(w)q_{no}(w'), \quad (4.67)$$

with $q_{ij}(w, w')$ defined in Eq. (2.76). The quantities we are mostly interested in the following are

$$\langle (\delta\kappa_{ij}^s)^2 \rangle = \langle (\kappa(w_i, 0; \hat{n}) - \kappa(w_j, 0; \hat{n}))^2 \rangle, \quad (4.68)$$

and

$$\langle (\kappa^s)^2 \rangle = \frac{9H_0^4 \Omega_{m,0}^2}{2(2\pi)^2} \int dw \int dw' q_{soso}(w, w') \int dk k^2 j_0(k(w - w')) P_\delta(k, w, w'). \quad (4.69)$$

(We remind the reader that the indices o,s denote observer, source respectively.) Analogous formulas hold for $\langle (\delta\kappa_{ij}^{ls})^2 \rangle$ and $\langle (\kappa^{ls})^2 \rangle, \langle (\kappa^l)^2 \rangle$. The line of sight integrals invoke the power spectrum of matter

System	σ_{lin}	σ_{halofit}	σ^{TDCOSMO}	$\kappa_{\text{ext}}^{\text{TDCOSMO}}$
DES0408-5354	0.0109	0.0390	0.0380	$-0.0397^{+0.0421}_{-0.0242}$
HE0435-1223	0.0088	0.0299	0.0342	$0.0040^{+0.0363}_{-0.0215}$
PG1115+080	0.0089	0.0303	0.0330	$-0.0054^{+0.0358}_{-0.0209}$
SDSS1206+4332	0.0092	0.0313	0.0410	$-0.0037^{+0.0402}_{-0.0215}$
B1608+656	0.0076	0.0251	0.0903	$0.1026^{+0.0949}_{-0.0451}$
RXJ1131-1231	0.0037	0.0110	0.0433	$0.0695^{+0.0480}_{-0.0260}$
WFI2033-4723	0.0087	0.0295	0.0660	$0.0591^{+0.0863}_{-0.0442}$

TABLE 4.1: Comparing our external convergence estimates with ray tracing results from the literature. We show the external convergence variance using linear theory (Eq. (1.70)) on the second column and non-linear approximation (Eq. (1.75)) on the third column. The ray tracing results from the TDCOSMO collaboration (available *here*) are shown in the last two columns (κ^s variance on the fourth, κ^s mean and the 16th and 86th percent quantiles on the fifth).

density perturbations δ , computed at non-equal times w, w' , which we will evaluate using the ansatz Eq. (1.75). A cutoff on the maximum k on the k integral must be

TDCOSMO derives Bayesian priors for external convergence by using ray-tracing through the Millenium simulation [Spr+05], on LOSs which are chosen to match the galaxy density observed in each strong lensing system of interest [Hil+09; Suy+10; Suy+13]. We can use these numerical results to compare with our analysis (Eq. (4.69)).

In Fig. 4.17 and Tab. 4.1 we compare our computation (linear, obtained using Eq. (1.70) for the power spectrum; and non-linear, obtained using Eq. (1.75)) with ray tracing results from the TDCOSMO collaboration available *here*. Fig. 4.18 shows the results obtained in Ref. [Suy+10] for the probability distribution of external convergence, averaging over all LOSs (that is, not restricting to fields containing strong lensing systems). Notice that we introduce a cut-off in the k integral at some k_{cutoff} . We choose $k_{\text{cutoff}} = 10 \text{ Mpc}^{-1}$, where individual galaxies and baryonic effects most likely lead to a breakdown of the HALOFIT result. Note that similar smoothing is implicitly used in the ray-tracing simulations when the gravitational potential is estimated from the distribution of matter. Changing k_{cutoff} by a factor of 2 up or down affects our results at the level of a few tens of percent, which is comparable to other theoretical uncertainties in our equations.

Our non-linear analysis (incorporating HALOFIT and the non-equal time approximation) reproduces the variance in κ^s to within about 30% accuracy for the systems which have a mean value of κ^s compatible with zero (first 4 systems in Tab. 4.1, top 4 panels in Fig. 4.17). Some systems, however, are found in [Suy+10] to be biased with a mean κ^s that is significantly off zero (last 3 systems in Tab. 4.1, bottom 3 panels in Fig. 4.17). This probably reflects excess structure along the LOS, typical of systems in crowded fields. For these systems, our calculation not only misses the bias, but also underestimates the spread in κ^s , by up to a factor of ~ 4 . Thus, indeed, the simplified computation from the previous section can only be used to provide a rough estimate of the magnitude of weak lensing effects, and ray tracing analyses on the lines of Refs. [Suy+10; Tih+18; Fas+06; Gre+13] are probably mandatory on a system by system study.

4.3.6 Examples of multiple source systems

A key point of our analysis is that the availability of multiple sources in a lensing system can only resolve the core-MSD degeneracy to the extent, that the core-induced term, $\kappa_{c0}(C_{i0} - C_{j0})$, is significantly larger than the natural expectation for the weak cosmological differential convergence term, $\delta\kappa_{ij}^s - \delta\kappa_{ij}^{\text{ls}}$, in Eq. (4.65).

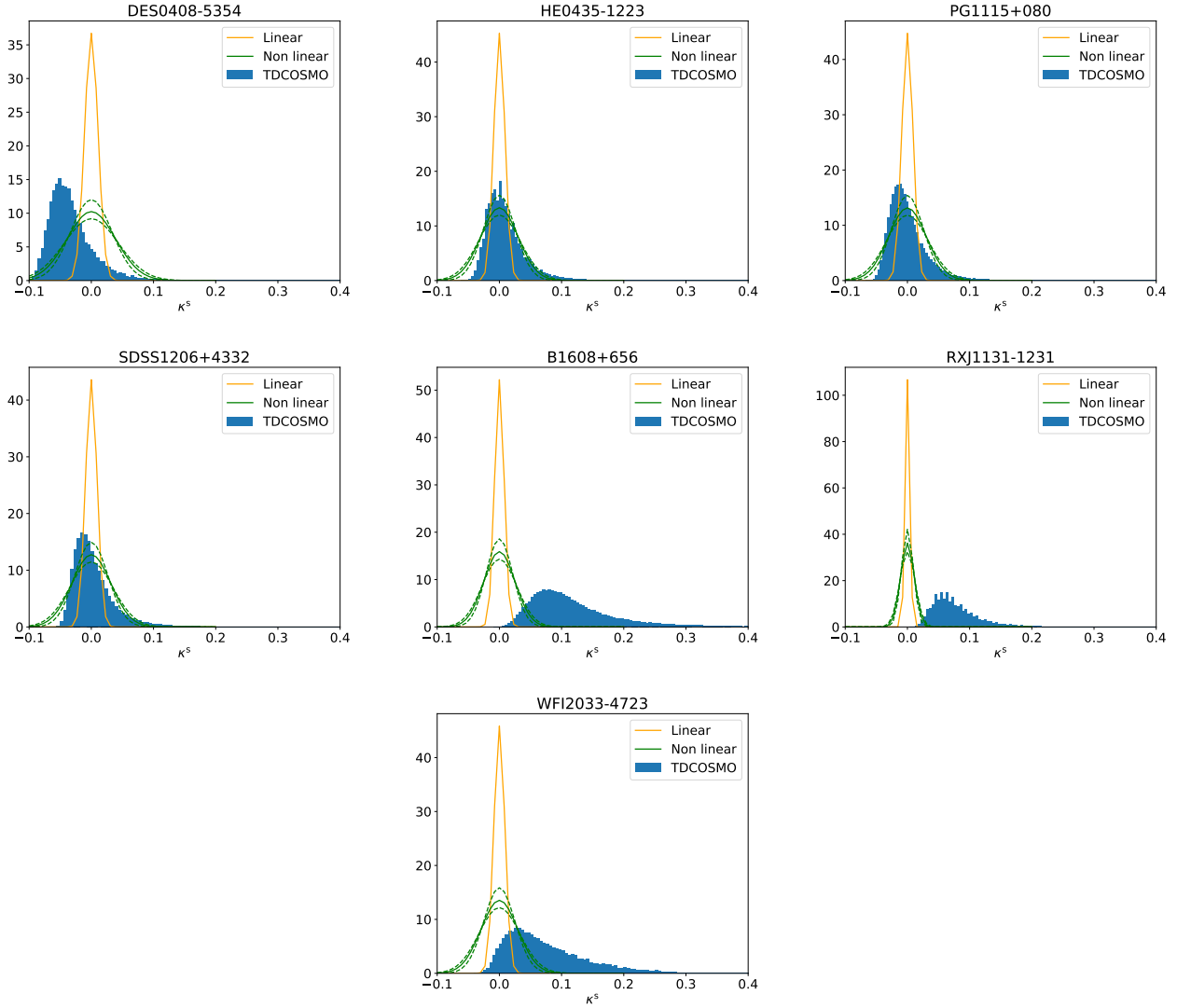


FIGURE 4.17: Comparing the probability distribution obtained in ray tracing [Suy+10] (blue bar histograms) with our computation, in linear theory (solid orange) and with the non-linear approximation (solid green: $k_{\text{cutoff}} = 10 \text{ Mpc}^{-1}$, dashed green: $k_{\text{cutoff}} = 5$ and 20 Mpc^{-1}). We remark that our results cannot reproduce the bias on the external convergence (nonzero mean seen in some of the blue bar histograms), since our computation is equivalent to an average over all LOSs, differently from the ray-tracing analysis TDCOSMO performs, which is calibrated to match the richness of the actual lensing systems. A fairer comparison between our computation and typical TDCOSMO results, obtained by averaging over many LOSs, is shown in Fig. 4.18. Code: .

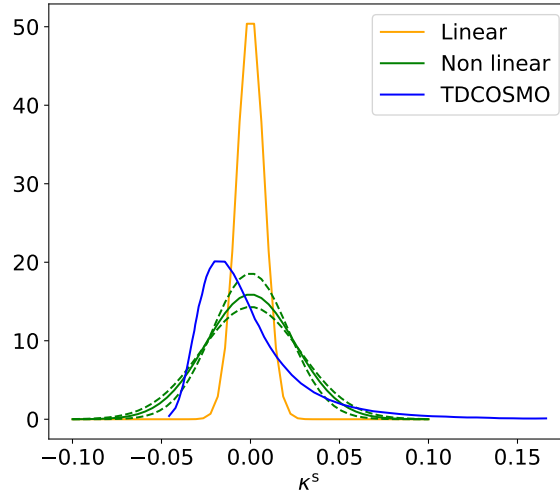


FIGURE 4.18: Comparing the distribution of the system B1608+656, when the average in the Millennium Simulation is done over all LOSs, with our estimates. Code: .

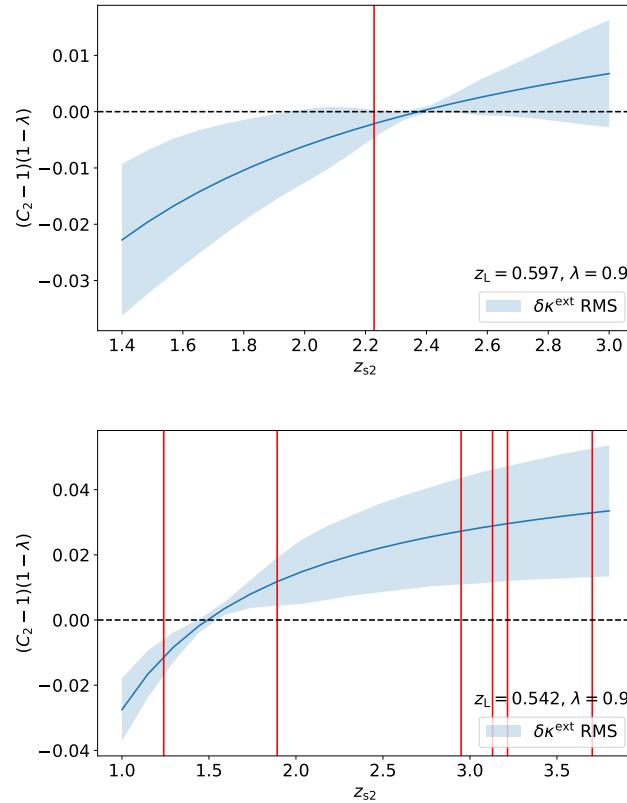


FIGURE 4.19: The angular diameter function $C_2 - 1$, $C_2 \equiv C_{20}$, weighted by a factor $\kappa_{c1} \equiv 1 - \lambda = 0.1$ (chosen to mimic a resolution of the lensing H_0 tension), compared with the cosmological RMS weak differential convergence $\delta\kappa$. **Top:** redshift parameters chosen to resemble the TDCOSMO system DESJ0408-5354 [Sha+20]. **Bottom:** parameters chosen to resemble the MACS J1149.5+2223 cluster system [Gri+18; Gri+20]. In both panels, the function $C_2 - 1$ vanishes at the redshift of the primary source (the source to which time-delays are measured). Vertical red lines mark the redshifts of secondary sources, that one could try to use to resolve the core-MSD. Code: .

DESJ0408-5354

As noted earlier, this galaxy lensing system has a primary lens at $z_1 \approx 0.6$, and main source (lensed quasar-host galaxy) at $z_1 \approx 2.3$, and a secondary source at $z_2 \approx 2.2$. A TDCOSMO analysis of this system, fitting an elliptic power-law density model for the lens (without allowing for a core component), inferred a value of H_0 which was $\approx 11\%$ higher than the CMB/LSS result [Sha+20]. Thus, a core component at $\kappa_{c0} \approx 0.1$ could completely resolve the lensing H_0 tension for this system [BCS20; BT21b].

The question arises, whether the presence of the second lensed source for this system could resolve the core-MSD associated with κ_{c0} . To address this question, in the **top** panel of Fig. 4.19 we plot (blue line) the function $C_{i0} - 1$ for this system, weighted by the factor $\kappa_{c0} \equiv 1 - \lambda = 0.1$. To demonstrate the confusion with weak external convergence, following Eq. (4.65) we superimpose a band with width chosen as the RMS value of $\delta\kappa_{i0}^s - \delta\kappa_{i0}^{ls}$ for the system. The red vertical line marks the redshift of the actual secondary source.

We conclude that multi-source imaging data for DESJ0408-5354 [Sha+20] is unlikely to help in constraining the core-MSD proposal sufficiently to solve the lensing H_0 tension.

MACS J1149.5+2223

As noted earlier, the lens in MACS J1149.5+2223 [Tre+16; Gri+18; Gri+20] is a galaxy cluster at $z_1 \approx 0.54$. The main source is a type-Ia supernova at $z_1 \approx 1.5$. Six additional multiply-imaged sources are distributed in redshift in the range $z_i \approx 1.2$ to $z_i \approx 3.7$.

In the **bottom** panel of Fig. 4.19 we show that for the secondary sources in MACS J1149.5+2223 [Gri+18; Gri+20], weak differential convergence should significantly (although, perhaps, not entirely) mask the presence of an internal MSD. We thus expect that adding differential convergence as nuisance parameters for the secondary sources (that is, the sources additional to the SNIa host, to which time delays were specified in the mock of [Gri+20]) would significantly increase the uncertainty on the impact of the MSD as compared to the preliminary results in the appendix of [Gri+20].

4.3.7 Summary of the multiple sources MSD with LSS

In this section we discussed some issues related to the MSD, as presented in Ref. [Teo+22]. Regarding the use of kinematics, Sec. 4.3.2, we noted that the relation between kinematics constraints and imaging data involves a combination of weak lensing terms that includes all of the observer-source, observer-lens, and source-lens segments of the LOS. Neglecting the source-lens and observer-lens convergence terms – a common practice in current analyses – could lead to a bias of the order of a few percent in the inference of H_0 from time delays. It is possible to account for the effect by adding the observer-lens term as nuisance parameter in the combined imaging+kinematics likelihood.

As we review in Sec. 4.3.1, the MSD is not broken by the availability of multiple sources in the imaging analyses. In Sec. 4.3.3 and Sec. 4.3.4 we considered what multiple sources do allow one to measure, which is differential convergence between different sources. Interestingly, weak differential external convergence complicates attempts to resolve the internal MSD, even if the internal core effect is parameterically larger than the weak lensing terms. The problem is that multiple sources are only useful against the internal MSD to the extent that they come with significantly different angular diameter distances; in practice, however, the angular diameter distances in typical multi-source systems used in cosmography are similar to the 10% level.

In Sec. 4.3.5 we used a non-perturbative calculation of cosmological external convergence, described in Sec. 2.4, that allows us to provide rough estimate of the expected size of the effect, as well as estimates of statistical correlations between different convergence terms. Our calculation suggests (what we think

is) a natural approximate way to account for non-linear matter power spectra entering in correlation functions at different values of the cosmic time variable.

Chapter 5

AxionH0graphy

As we explained in Sec. 4.1, a core-MSD could introduce an important bias in the strong gravitational lensing measurement of H_0 .

In Refs. [BT21b; BT25] we consider the possibility that physical realizations of cores entering into the core-MSD may come from ULDM, in the form of the solitons discussed in Sec. 3.2. However, if instead of using time delays to measure H_0 , one assumes a H_0 prior, then time-delays can be used to probe the presence of such ULDM cores. We call such a way to look for ULDM cores AxionH0graphy. We show that time-delay cosmography can be used to put constraints on the ULDM paradigm in the mass range $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$ for cosmological ULDM to DM fractions $\alpha_\chi \lesssim 0.15$.

In Sec.5.1, we explain to which extent ULDM cores can work as an approximate MSD; in Sec. 5.2, we see how ULDM cores would affect the H_0 prediction for TDCOSMO [Mil+20] systems; in Sec. 5.3 we illustrate our main point regarding the AxionH0graphy effect via a mock system; in Sec. 5.4 we perform numerical simulations to check the predictions based on the theoretical expectations of Sec. 5.2.1; we summarize in Sec. 5.5.

5.1 ULDM cores as approximate MSD

As mentioned in Sec. 4.1, there are two conditions that a ULDM core should satisfy to behave as an approximate MSD and give a order 10% shift on the inferred H_0 : imaging error due to the approximate nature of the core-MSD should be small enough, and a big enough central convergence. We thus need to require

$$\delta_E \lesssim 0.01, \quad (5.1)$$

$$1 - \lambda = \kappa_c(\theta_E) \sim 0.1 = \frac{\delta H_0}{H_0}, \quad (5.2)$$

where δ_E was defined in Eq. (4.3). These translate into, using the properties of the soliton profile Eq. (3.17)

$$m \lesssim 10^{-24} \text{ eV} \left(\frac{1''}{\theta_E} \right)^{3/2} \left(\frac{1 \text{ Gpc}}{D_1} \right) \left(\frac{\delta H_0/H_0}{0.1} \right)^{-1/2}. \quad (5.3)$$

These estimates make us focus on the $\mathcal{O}(0.1)$ ULDM fraction on the mass range $m \lesssim 10^{-24} \text{ eV}$. In particular, we will need to consider a soliton in the presence of an external gravitational potential Φ_{ext} . The other $\mathcal{O}(0.9)$ DM fraction can be thought as standard CDM, or for definiteness just a $m \gg 10^{-24} \text{ eV}$ axion.

Fig. 5.1 shows a typical setting of a massive elliptical (the lens galaxy) with a subdominant ULDM component, which forms a cored structure which is large enough to act as a core-MSD, and big enough

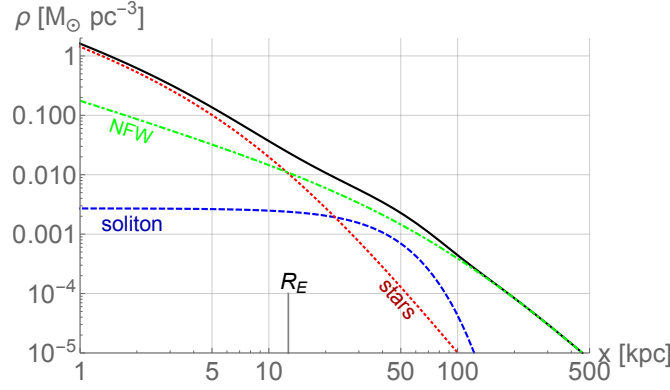


FIGURE 5.1: ULDM soliton core as a possible 10% H_0 bias on the gravitational lensing measurement. Green dash-dotted and red dotted lines show the density profiles of a cold dark matter NFW and a stellar component, respectively. The blue dashed line shows a soliton with $m = 2 \times 10^{-25}$ eV and $M \approx 1.4 \times 10^{12} M_\odot$, producing a shift $\delta H_0/H_0 \approx 0.1$. Such a soliton comes about from an ULDM component with a cosmological fraction of order 10% with respect to the whole dark matter budget. The halo parameters resemble the TDCOSMO system DESJ0408 [Sha+20].

to give a 10% shift on the inferred H_0 . Parameters are chosen to resemble the TDCOSMO system DESJ0408 [Sha+20].

A ULDM soliton could produce the κ_c term in Eq. (4.1). It is convenient to introduce an approximation with which properties of the soliton can be derived analytically. We choose

$$\chi_1(r) \approx \frac{1}{(1 + a^2 r^2)^b}, \quad (5.4)$$

where the coefficients a and b are fitted numerically to the exact solution. For a self-gravitating soliton (the limit $\Phi_{\text{ext}} \rightarrow 0$), we obtain $a \approx 0.23$ and $b \approx 3.9$. When $\Phi_{\text{ext}} \neq 0$, the coefficients a and b depend on λ_s and Φ_{ext} via the combination $\lambda_s^{-2} \Phi_{\text{ext}} (\lambda_s^{-1} r)$, as detailed in Eq. (3.20).

With the approximation of Eq. (5.4), the soliton mass is

$$M_\lambda = \frac{\lambda_s}{Gm} \int dr r^2 \chi_1^2(r) \approx \frac{\lambda_s}{Gm} \frac{\sqrt{\pi} \Gamma(2b - \frac{3}{2})}{a^3 4\Gamma(2b)}. \quad (5.5)$$

The convergence, displacement angle, and lensing potential are:

$$\kappa_\lambda(\theta) \approx \frac{\lambda_s^3 m}{4\pi G \Sigma_{\text{cr}}} \frac{\sqrt{\pi} \Gamma(2b - \frac{1}{2})}{a \Gamma(2b)} \frac{1}{\left(1 + \frac{\theta^2}{\theta_c^2}\right)^{2b - \frac{1}{2}}}, \quad (5.6)$$

$$\alpha_\lambda(\theta) \approx \kappa_\lambda(0) \frac{2\theta_c^2}{(4b - 3)\theta} \left(1 - \left(1 + \frac{\theta^2}{\theta_c^2}\right)^{\frac{3}{2} - 2b}\right), \quad (5.7)$$

$$\psi_\lambda(\theta) \approx \kappa_\lambda(0) \frac{\theta^2}{2} {}_3F_2 \left[\left\{1, 1, 2b - \frac{1}{2}\right\}, \left\{2, 2\right\}; -\frac{\theta^2}{\theta_c^2} \right], \quad (5.8)$$

where we defined the core angle

$$\theta_c = \frac{1}{\lambda_s a m D_l}, \quad (5.9)$$

and where ${}_p\mathcal{F}_q \left[\vec{a}, \vec{b}; z \right]$ is the generalised hypergeometric function¹. In the MSD limit, $\theta_c \gg \theta$, one can verify that $\kappa_\lambda(\theta) \approx \kappa_\lambda(0)$, $\alpha_\lambda(\theta) \approx \kappa_\lambda(0)\theta$, and $\psi_\lambda \approx \kappa_\lambda(0)\theta^2/2$.

Adopting the soliton as our core-MSD component, we set $\kappa_c(\theta) \equiv \kappa_\lambda(\theta)$ in Eq. (4.1). From Eqs. (4.3) and (4.4) we get

$$\frac{\delta H_0}{H_0} \approx \frac{\kappa_\lambda(0)}{\left(1 + \frac{\theta_E^2}{\theta_c^2}\right)^{2b-\frac{1}{2}}}, \quad (5.10)$$

and

$$\delta_E \approx \frac{\left(1 + \frac{\theta_E^2}{\theta_c^2}\right)^{2b-\frac{1}{2}} - (2b - \frac{1}{2}) \frac{\theta_E^2}{\theta_c^2} - 1}{(2b - \frac{3}{2}) \frac{\theta_E^2}{\theta_c^2}} \frac{\delta H_0}{H_0}. \quad (5.11)$$

For $\theta_E \ll \theta_c$ we have $\delta_E \approx (b - \frac{1}{4}) \frac{\theta_E^2}{\theta_c^2} \frac{\delta H_0}{H_0}$. This shows how imaging uncertainties, roughly summarised by δ_E , constrain the shift $\delta H_0/H_0$ at given soliton core angle θ_c .

In next section we calculate how the soliton profile is distorted in the presence of a PL background. We find that Eq. (5.4) is still a good approximation, sufficient for our needs; the effect of the background density is to modify the numerical values of the coefficients a and b .

We comment on the parameter space of the model. As noted in the beginning of this section, at a fixed value of the ULDM particle mass m , the soliton is a single-parameter function. While the scaling parameter λ_s from Eq. (3.19) is convenient for analytical expressions, in making contact with observations we prefer to use the total soliton mass M , substituting $\lambda_s \rightarrow \lambda_s(M, m)$ using Eq. (5.5). (The detailed matching, but not the basic procedure, is slightly modified with a background potential as explained in Sec. 5.1.1.) All other properties of the core (the convergence, for example) then depend only on m and M . The full parameter space is therefore covered when we analyse our results in terms of m and M .

5.1.1 Power law background

Here we consider how an external mass distribution affects the soliton profile written in the form of Eq. (5.4). The background is taken to be a pure PL. Lensing analyses have often adopted this approximation, which leads to results for H_0 that are consistent with more realistic composite DM+stars halo models [Mil+20]. In a realistic analysis, the halo is axi-symmetric to accommodate quad geodesics, and we include axi-symmetry when we analyse mock data in Sec. 5.3. For simplicity, however, in modelling the impact of the external potential on the structure of the soliton we assume spherical symmetry. This approximation is justified by the disk galaxy study of Ref. [Bar+19], which showed that the soliton remains nearly spherical even with significant a-sphericity of the background.

To simplify matters further we set $\gamma = 2$ in Eq. (2.41), close to the slopes inferred for the galaxies in Tab. 5.1. The external potential is then given by:

$$\Phi_{\text{ext}}(r) = A \ln r + \text{const}, \quad (5.12)$$

$$A = 2G\Sigma_{\text{cr}}D_1\theta_E. \quad (5.13)$$

Note that the factor $\Sigma_{\text{cr}}D_1 = D_s/(4\pi G D_{1s})$ is independent of H_0 . To gain some physical intuition, note that if we define $M_{\text{PL}}(1/m)$ as the mass included in the PL profile up to a distance equal to the ULDM Compton radius $1/m$, then $A = GM_{\text{PL}}(1/m)m$. Conveniently, for $\gamma = 2$, $M_{\text{PL}}(1/m)m$ is independent on m .

¹A rapidly converging expression is ${}_3\mathcal{F}_2 \left[\left\{1, 1, 2b - \frac{1}{2}\right\}, \{2, 2\}; z \right] = \frac{2}{z(3-4b)} \left(\log(1-z) - \sum_{n=1}^{\infty} \frac{\Gamma(\frac{5}{2}-2b+n)}{\Gamma(\frac{5}{2}-2b)} \frac{z^n}{(z-1)^n n!} \right)$.

Because Φ_{ext} breaks the scale invariance of the self-gravitating soliton, the coefficients a and b in the approximation of Eq. (5.4) now depend on the combination A/λ_s^2 . We find that Eq. (5.4) still provides a good fit for any value of A/λ_s^2 , with the fitting formula:

$$a(z) = 0.23 \sqrt{1 + 7.5z \tanh(1.5z^{0.24})}, \quad (5.14)$$

$$b(z) = 1.69 + \frac{2.23}{(1 + 2.2z)^{2.47}}, \quad (5.15)$$

where $z = A/\lambda_s^2$.

5.2 Constraints on ULDM from TDCOSMO systems

We are ready for a rough assessment of the lensing H_0 tension in the ULDM model. In Fig. 5.2 we explore $\delta H_0/H_0$ and δ_E as function of the ULDM particle mass m (x -axis) and soliton mass M (y -axis). The different panels correspond to the different systems in Tab. 5.1, systems which feature in the TDCOSMO campaign [Mil+20]. The information in the plot is as follows.

We begin with results that include the effect of a background (non-ULDM) external potential, modelled as a pure PL, using the results in Sec. 5.1.1. For simplicity, the same PL index $\gamma = 2$ is used for all systems, but the value of θ_E and δ_E for each system is as in Tab. 5.1. In the pale red shaded region, δ_E exceeds its corresponding value from Tab. 5.1. This region is disfavoured by the imaging data. Along the blue dashed line, $\delta H_0/H_0$ matches the value required to solve the H_0 tension. The solid blue lines delimit the uncertainty on $\delta H_0/H_0$ for each system. (Other curves in Fig. 5.2 correspond to theoretical constraints and are explained later.)

We also show how the imaging constraints change if the external PL density is not included in the soliton computation. The result is shown by the dark red shaded region. The imaging constraint is generally weaker when the PL effect is not included, compared to when it is (i.e., the dark shaded region is contained inside the pale region), because the background potential causes the soliton to contract inwards at fixed m and M , decreasing θ_c and leading to stronger violation of the MSD.

The fact that the PL background analysis provides stronger imaging constraints, compared with the self-gravity case, illustrates the sensitivity of the analysis to the detailed mass profile of the lens. However, the soliton contraction is mostly driven by the cuspy PL mass distribution at small $r \ll R_E$, where the lensing observables are not well constrained. In fact, the observed stellar surface brightness of the lenses display cores rather than cusps on distances $r \ll R_E$, where the stellar density dominates over the DM. As a result, physically-motivated composite stellar+DM halo models, adjusted to fit the stellar light profile, predict a contraction effect on the soliton that is less significant than in the PL background. The imaging constraint in these more realistic background models are closer to the self-gravitating soliton result.

In Fig. 5.1 we show a soliton solution of the lensing H_0 tension, using a composite stellar+DM model that mimics the properties of the system DESJ0408. The solution has $m = 2 \times 10^{-25}$ eV and $M \approx 1.4 \times 10^{12} M_\odot$, marked in Fig. 5.2 by a circle (bottom-left panel). This solution has $\delta H_0/H_0 \approx 0.1$ and we have verified that it is compatible with the requirement $|\delta_E| < 0.01$, valid for this system. The fact that this solution would seem to be excluded in the PL background analysis is due to the exaggerated soliton contraction in the PL case.

We can use the self-gravitating soliton case to understand the imaging constraints parametrically. In this limit $a \approx 0.23$ and $b \approx 3.9$ can be used in Eqs. (5.5-5.11) and the H_0 shift is

$$\frac{\delta H_0}{H_0} \approx 2.9 \frac{\lambda_s^3 m}{4\pi G \Sigma_{\text{cr}}}. \quad (5.16)$$

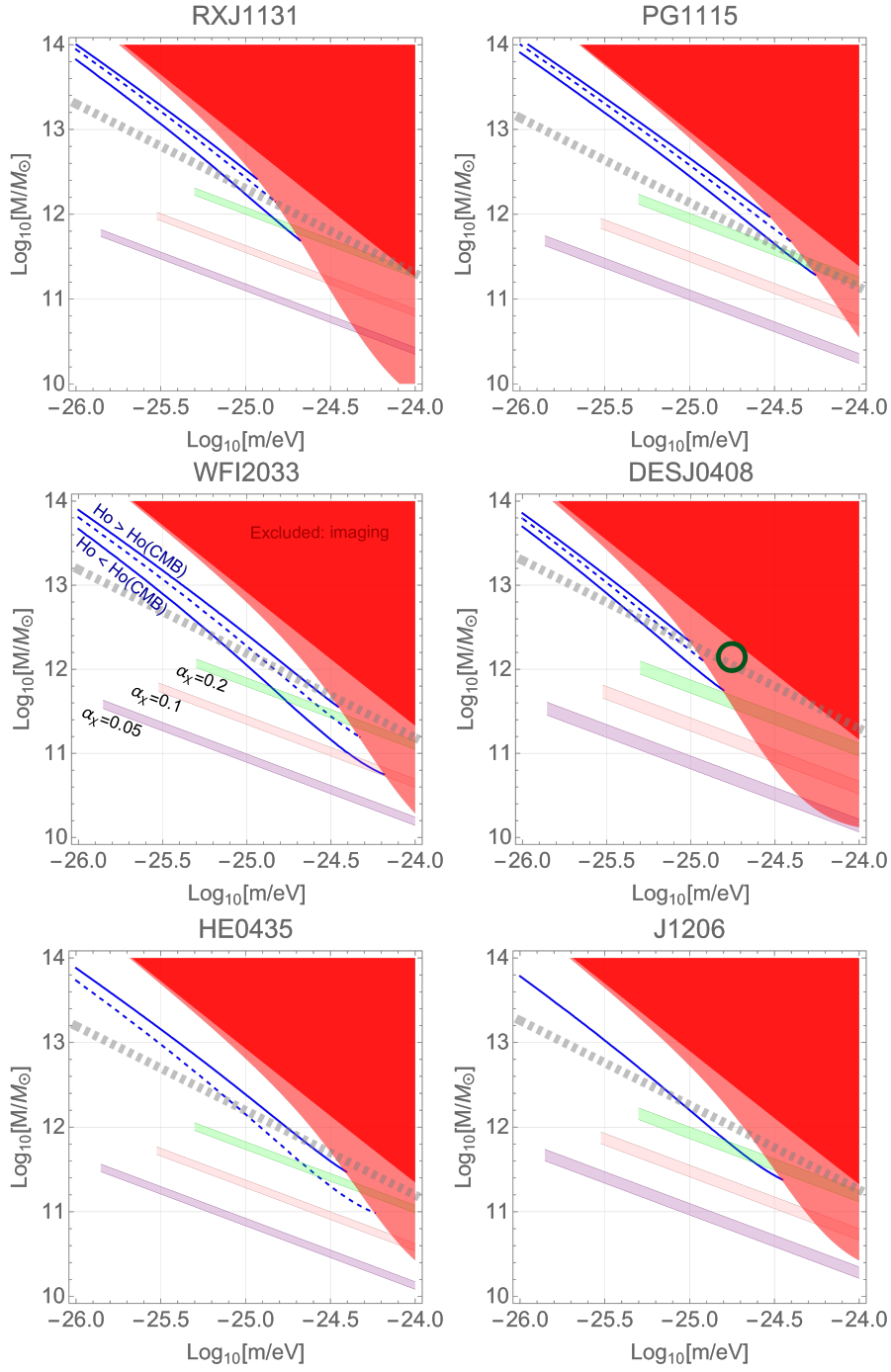


FIGURE 5.2: ULDM soliton as a bias to the lensing H_0 measurement. Panels correspond to different systems in Tab. 5.1. The dark red shaded region shows where δ_E exceeds its limits from Tab. 5.1, computed neglecting the effect of an external mass distribution on the soliton. The pale shaded region shows the same constraint, including the effect of an external PL potential. Along the blue dashed line, $\delta H_0/H_0$ matches the value from Tab. 5.1, with solid lines delimiting the uncertainty; this result also includes the PL background. (In J1206 and HE0435 the central value and/or lower limit of $\delta H_0/H_0$ are compatible with zero.) The relaxation estimate of Eq. (5.23) is shown by green, pink, and purple bands for $\alpha_{\chi} = 0.2, 0.1, 0.05$, respectively, using $\sigma_0 = \langle \sigma_{\text{los}} \rangle$. The band width is defined by the uncertainty in $\langle \sigma_{\text{los}} \rangle$. We truncate each constant- α_{χ} band at small m according to the cosmological constraints from Ref. [Lag+21]. The saturation estimate $[(K/M)_{\lambda} < 1.5(A/2)]$ is shown by thick dashed grey line. The circle in the panel of DESJ0408 marks the set-up of Fig. 5.1.

On the other hand, in the same self-gravity limit we have $\delta_E \approx 3.6 (\theta_E/\theta_c)^2 \delta H_0/H_0$. Demanding $\delta_E \lesssim 0.01$, as in typical systems, and setting $\delta H_0/H_0 \approx 0.1$, we should impose $\theta_c \gtrsim 6\theta_E$, or

$$\lambda_s m \lesssim \frac{0.7}{D_l \theta_E}. \quad (5.17)$$

Combining Eqs. (5.16) and (5.17) we obtain,

$$\text{self-gravity approximation : } m \lesssim 10^{-24} \text{ eV} \left(\frac{D_{ls}}{D_s} \right)^{\frac{1}{2}} \left(\frac{1''}{\theta_E} \right)^{\frac{3}{2}} \left(\frac{1 \text{ Gpc}}{D_l} \right) \left(\frac{0.1}{\delta H_0/H_0} \right)^{\frac{1}{2}}. \quad (5.18)$$

Again, the presence of an external potential (PL or composite) contracts the soliton inward to some extent at fixed M and m , shifting the upper limit of Eq. (5.18) to somewhat lower m .

5.2.1 How much mass in a soliton?

To explain the lensing H_0 tension, the ULDM soliton mass in the lens galaxy must be large enough. How much ULDM is needed, and how does this requirement compare to the soliton predicted by numerical and analytic considerations?

Numerical simulations have shown that the soliton grows by accreting ULDM from the surrounding halo via gravitational dynamical relaxation, with a characteristic timescale

$$\tau_{\text{rel}} \approx \frac{\sqrt{2}b}{12\pi^3} \frac{m^3 \sigma^6}{G^2 \rho_\chi^2 \Lambda_C}. \quad (5.19)$$

Here, ρ_χ is the density of ULDM, σ is the velocity dispersion, Λ_C is the Coulomb logarithm, and the numerical factor $b \approx 0.7$ was calibrated in numerical simulations [LPT18] (see also [EN19; Che+21; Sch+20]). Below we will set $\sqrt{2}b \approx 1$.

A first estimate of the maximal mass of a ULDM soliton that could form in a galaxy can be obtained by calculating the ULDM mass contained inside the galactocentric radius R_g within which $\tau_{\text{rel}}(R_g) < t_{\text{gal}}$, where t_{gal} is the age of the galaxy. Near this radial boundary we expect that $\rho_\chi \approx \alpha_\chi \rho_m = \alpha_\chi \rho_0 / (1 - \alpha_\chi)$, where ρ_m is the total DM density (ULDM+non-ULDM) and ρ_0 is the background density in non-ULDM DM (α_χ is the cosmological ULDM fraction). We can thus estimate R_g from solving

$$t_{\text{gal}} \approx \frac{1}{12\pi^3} \frac{m^3 \sigma^6(R_g)}{G^2 \Lambda_C^2 \rho^2(R_g)}. \quad (5.20)$$

A rough upper bound on the mass of a soliton is then

$$M < \alpha_\chi M_{\text{halo}}(R_g). \quad (5.21)$$

For an isothermal power-law halo with $\rho \propto R^{-2}$ and constant $\sigma \approx \sigma_0$, we have $\sigma_0^2 \approx c_0 GM(R)/R \approx c_0 4\pi G \rho(R) R^2$ where we expect² $c_0 \approx 1/2$. With this we have

$$R_g^4 \approx 12\pi^3 \frac{G^2 \Lambda_C \alpha_\chi^2 \rho^2(R) R^4}{m^3 \sigma_0^6} t_{\text{gal}} \approx \frac{3\pi}{4} \frac{\Lambda_C \alpha_\chi^2}{c_0^2 m^3 \sigma_0^2} t_{\text{gal}}. \quad (5.22)$$

²See [BT08], Ch.4.3. We keep track of the constant c_0 here because in a realistic scenario it could vary by $\mathcal{O}(1)$, contributing to the uncertainty in the relaxation estimate.

On the other hand, $M(R_g) \approx \sigma_0^2 R_g / (c_0 G)$, so using Eq. (5.21) the soliton upper bound reads

$$M < \left(\frac{\alpha_\chi^2}{m} \right)^{3/4} \frac{1}{G} \left(\frac{\sigma_0}{c_0} \right)^{3/2} \left(\frac{3\pi}{4} \Lambda_C t_{\text{gal}} \right)^{1/4}. \quad (5.23)$$

In Fig. 5.2 we show how the estimate of Eq. (5.23) compare with the imaging and H_0 constraints. The upper bound is shown by the green, pink, and purple bands, corresponding to $\alpha_\chi = 0.2, 0.1$, and 0.05 , respectively. The upper and lower limits of each of the bands are obtained by setting $\sigma_0 = \langle \sigma_{\text{los}} \rangle$ in Eq. (5.23) and using the upper and lower uncertainty estimates for $\langle \sigma_{\text{los}} \rangle$ from Tab. 5.1. The age of each lens galaxy (t_{gal} in Eq. (5.23)) is taken as the FLRW time between $z = 20$ and the lens redshift z_1 .

We truncate each constant- α_χ band at small m according to the cosmological constraints from Ref. [Lag+21]. We also adhere, roughly, to the limit of [Kob+17] by restricting to $\alpha_\chi \leq 0.2$. Inspecting the result, it is clear that the cosmological constraints on α_χ play an important role in the scenario. While the imaging constraints eliminate $m \gtrsim 10^{-24}$ eV or so, the combination of the dynamical relaxation consideration with the cosmological bounds [Kob+17; Lag+21] disfavors $m \lesssim 10^{-25}$. This defines the interesting parameter space of the model to a rather narrow window.

Apart from the dynamical relaxation upper bound, another consideration comes from the saturation of the growth of the soliton: while Eq. (5.23) estimates the maximal amount of ULDM mass that is available for condensation into a soliton, it is possible that only a fraction of this total available mass would actually condense. The soliton growth slows from $M \propto (t/\tau_{\text{rel}})^{1/2}$ to $M \propto (t/\tau_{\text{rel}})^{1/8}$ when the specific kinetic energy of the soliton (kinetic energy per unit mass, (E_{kin}/M)) becomes comparable to the specific kinetic energy in the surrounding halo. Both the $M \propto t^{1/2}$ growth phase and its saturation into $M \propto t^{1/8}$ were observed in numerical simulations [EN19; Che+21; Sch+20], and are consistent with the soliton–host halo relation originally discovered in [SCB14; Sch+14], and then shown to be equivalent to (E_{kin}/M) equilibration in [Bar+18; Bar+19]. The reason for this saturation is that once the (E_{kin}/M) threshold is crossed, the velocity dispersion at the outskirts of the soliton, and thus the dynamical time scale τ_{rel} , becomes dominated by the gravitational potential of the soliton itself. This causes τ_{rel} to depend on M with larger M corresponding to larger τ_{rel} , leading to self-regulation of the growth rate.

With the parameterization of Eq. (5.4) we can compute the soliton specific kinetic energy,

$$\left(\frac{E_{\text{kin}}}{M} \right)_\lambda = \lambda_s^2 \frac{\int dr r^2 (\partial_r \chi_1)^2}{2 \int dr r^2 \chi_1^2} \approx \lambda_s^2 \frac{3a^2 b^2 \Gamma(2b) \Gamma(2b - \frac{1}{2})}{\Gamma(2b + 2) \Gamma(2b - \frac{3}{2})}. \quad (5.24)$$

In the limit of low-mass soliton, where the background gravitational potential completely dominates the structure of χ , $(E_{\text{kin}}/M)_\lambda$ is independent of the parameter λ_s because the ULDM profile simply reflects the wave function of an ULDM particle bound in the external potential. Indeed, using the PL external potential in this limit gives $(E_{\text{kin}}/M)_\lambda \approx A/2$, consistent with the virial theorem³. We can estimate the self-regulation threshold by letting the soliton mass grow until $(E_{\text{kin}}/M)_\lambda$ starts to exceed the background-dominated result.

In Fig. 5.3 we illustrate the growth saturation limit, computed for the system RXJ1131 from Tab. 5.1. On the x-axis we plot $(E_{\text{kin}}/M)_\lambda$ normalised to its asymptotic small- M value. On the y-axis we plot the product $M_\lambda m_{25}$, where m_{25} corresponds to the ULDM particle mass via $m_{25} = m / (10^{-25} \text{ eV})$. As noted above, at small M_λ the value of $(E_{\text{kin}}/M)_\lambda$ becomes independent of M_λ (or equivalently, of λ). As M_λ increases, the soliton self-gravity begins to dominate $(E_{\text{kin}}/M)_\lambda$. In Fig. 5.3, we mark by a red dot the value of M_λ at which $(E_{\text{kin}}/M)_\lambda$ exceeds the small- M result by 50%. From Eq. (5.5), we know that in the

³To be precise, the large- A/λ_s^2 limit of Eq. (5.24) gives $(E_{\text{kin}}/M)_\lambda \rightarrow 0.454A$. The small mismatch from $1/2$ can be expected given that Eq. (5.4) is merely an approximation for the soliton.

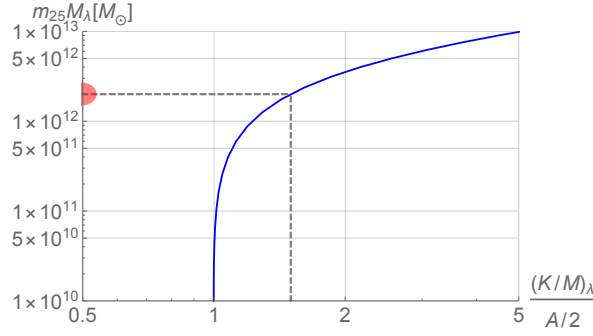


FIGURE 5.3: Illustration of the soliton growth threshold, $K \equiv E_{\text{kin}}$ (see text below Eq. (5.24)).

self-gravitation limit the parameter λ_s fixes the combination $M_{\lambda} m$; thus, the saturation limit also fixes the combination $M_{\lambda} m$. This is the reason why we use the product $M_{\lambda} m_{25}$ for the y-axis in Fig. 5.3.

With some arbitrariness, we will estimate the growth saturation limit (roughly) by imposing, for each halo, $(E_{\text{kin}}/M)_{\lambda} < 1.5(A/2)$, similar to the illustration in Fig. 5.3. The result of this calculation is shown by the thick dashed grey lines in Fig. 5.2.

For all of the systems of Tab. 5.1, the growth saturation limit is weaker than or comparable to the dynamical relaxation time-scale constraint obtained with ULDM fraction $\alpha_{\chi} = 0.2$. This suggests that for the range of m plotted in Fig. 5.2, ULDM solitons are still growing in the lens galaxies, and the limiting factor for the soliton mass may be the total ULDM mass available within the dynamically relaxed region of the halo. In Sec. 5.4, we perform dedicated simulations to test the estimates of the soliton mass arising in (somewhat) realistic halos, confirming (with caveats discussed there) the AxionH0graphy effect.

Tab. 5.1 shows the values of $\delta H_0/H_0$ required to bring the different systems to accord with the CMB result. We see that $\kappa_c(\theta_E) \approx 0.1$, with some variation between systems [Suy+17; Bon+19; Bir+19; Che+19; Won+20], could have a large enough bias to bring the lensing H_0 measurement back to the CMB value.

TABLE 5.1: Lens systems from [Mil+20]. Values for H_0 (in km/s/Mpc) are from the PL fit (Fig. 6 in [Mil+20]). The reference “true” H_0 used to define $\delta H_0/H_0$ is taken from the CMB result $H_0 = 67.36 \pm 0.54$ km/s/Mpc [Agh+20]. θ_E is in arcsec. $\langle \sigma_{\text{los}} \rangle$ is in km/s. On the last column we show twice the maximum relative error of the velocity anisotropy, useful for comparison with $\delta H_0/H_0$.

	$\delta H_0/H_0$	γ	θ_E	δ_E	z_l	z_s	σ^P	$2 \delta \langle \sigma_{\text{los}} \rangle / \langle \sigma_{\text{los}} \rangle$
RXJ1131	$0.13^{+0.05}_{-0.06}$	1.98	1.6	0.006	0.295	0.654	320 ± 20	0.125
PG1115	$0.23^{+0.11}_{-0.10}$	2.18	1.1	0.02	0.311	1.722	280 ± 25	0.178
HE0435	$0.06^{+0.07}_{-0.07}$	1.87	1.2	0.025	0.4546	1.693	220 ± 15	0.136
DESJ0408	$0.11^{+0.04}_{-0.04}$	2	1.9	0.01	0.597	2.375	230 ± 27	0.235
WFI2033	$0.08^{+0.05}_{-0.04}$	1.95	0.9	0.016	0.6575	1.662	250 ± 19	0.152
J1206	$-0.01^{+0.08}_{-0.07}$	1.95	1.2	0.025	0.745	1.789	290 ± 30	0.207

5.3 Mock analysis

Our discussion of the imaging constraints was simplistic, in that we used the rough Einstein angle criterion δ_E to constrain the possible shift to H_0 . In comparison, the likelihood function in real lensing

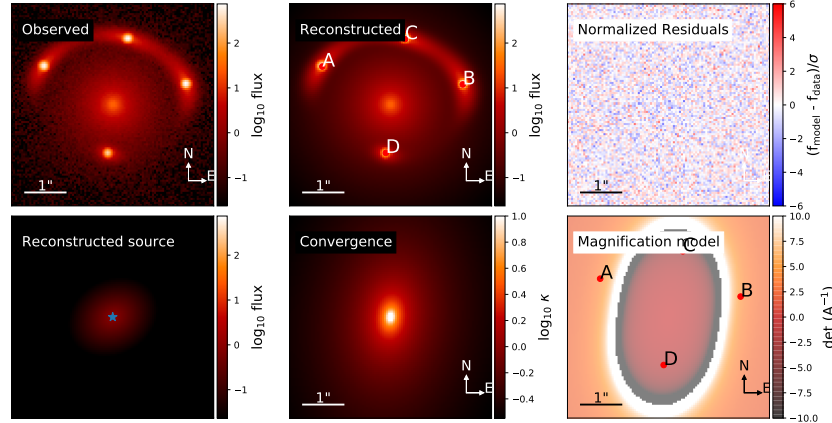


FIGURE 5.4: Mock image and reconstruction. The model used for inference is PL + ULDM core, with a Gaussian prior of $H_0 = 67.4$ km/s/Mpc. Code: $\text{\textcircled{Q}}$.

analyses contains detailed extended source information as well as multiple modelling parameters, experimental seeing limitations, etc. We here present a numerical study of a mock system, including most of these complications, using the gravitational lens model software package `lenstronomy` $\text{\textcircled{Q}}$ [BA18b; BAR15; Bir+21]. This numerical study serves two purposes. First, we introduce an implementation of the ULDM module in `lenstronomy`, as detailed in App. D.1. Second, this exercise allows us to test the accuracy of the simple δ_E criterion. We find that the naive δ_E criterion is slightly conservative compared with a full analysis: for example, at fixed $\delta H_0/H_0 \approx 0.1$, we find that a full numerical analysis yields a constraint on θ_c (and therefore, equivalently, on m at fixed M) that is about a factor of 2 weaker than the constraint we would obtain using the naive δ_E criterion.

The mock data is as follows. The truth model has the convergence of Eq. (4.1), with κ_0 given by an elliptic PL profile (so as to produce a quad image) and $\kappa_c = \kappa_\lambda$ of a ULDM soliton with $m = 10^{-25}$ eV and $M = 1.4 \times 10^{12} M_\odot$. The parameters are chosen to produce an effective $\kappa_\lambda(\theta_E) \approx 0.1$ and $\theta_c \approx 10''$. The truth value of H_0 is set to $H_0 = 67.4$ km/s/Mpc, mimicking the CMB result [Agh+20]. In figure 5.4 we show the mock alongside a reconstructed image, done by running the MCMC using the core-MSD model with a Gaussian prior on H_0 set at its truth value.

To demonstrate the outcome of using an inference model which does not include a core component (the case of, e.g. [Suy+17; Bon+19; Bir+19; Che+19; Won+20; Mil+20]), we run the MCMC using a pure (elliptic) PL. Fig. 5.5 shows the posterior triangle plot obtained for this model. As expected, the MCMC converges to $H_0 \approx 75$ km/s/Mpc, in a good fit without detectable imaging residuals. A lensing analysis that does not utilise the core-MSD model would converge to this biased result.

In the top panel of Fig. 5.6 we re-run the MCMC, this time using the core-MSD model in the inference. (For convenience in the implementation, we use $1/\theta_c$ and $\kappa_\lambda(\theta_E)$, rather than m and M , as the sampling parameters in the fit.) The MSD leads to a significant broadening of the H_0 posterior, corresponding to the $\kappa_\lambda(\theta_E)$ - H_0 degeneracy. A low value of $H_0 \approx 60$ km/s/Mpc, accompanied by an $M \approx 10^{12} M_\odot$ soliton at $m \lesssim 10^{-25}$ eV, produces a comparably good fit as the original $H_0 \approx 75$ km/s/Mpc model with a vanishing soliton (Fig. 5.5).

In the bottom panel of Fig. 5.6 we repeat the exercise, this time adding an external CMB prior on H_0 coincident with the truth value of the mock. The posterior now converges to an upper limit of $1/\theta_c \simeq 0.13/(1'')$ at 95% confidence level. This, together with the most probable value for $\kappa_\lambda(\theta_E)$, correspond to $M \approx 9 \times 10^{10} M_\odot$ and $m \approx 2 \times 10^{-25}$ eV.

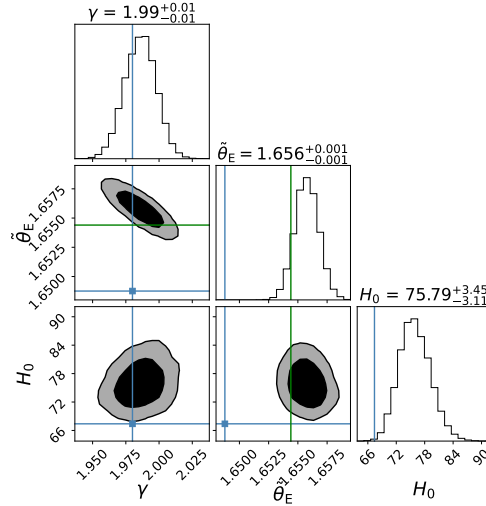


FIGURE 5.5: Lensing reconstruction and time delay analysis using mock data. Blue lines mark the truth values. The truth model is PL background + ULDM core, with $\kappa_\lambda(\theta_E) \approx 0.1$. The truth value of H_0 used to produce the time delays is $H_{0,\text{true}} = 67.4$ km/s/Mpc. The model used in the inference is pure PL, without a core. The fit parameters are the PL slope γ , the Einstein angle θ_E , and the halo ellipticity q (not shown in the plot). The PL fit converges on a false result $H_{0,\text{inferred}} \approx 75$ km/s/Mpc. Notice that this PL fit will try to converge to the true Einstein angle of the mock, which can be derived by solving $\theta_E = (1 - \kappa_\lambda(\theta_E))\tilde{\theta}_E + \alpha_c(\theta_E)$ where $\tilde{\theta}_E$ is the parameter we used to construct the mock; in green we show this θ_E . Code: [🔗](#).

To study how well Eq. (5.11) approximates realistic imaging constraints on the soliton, in Fig. 5.7 we show δ_E as a function of $1/\theta_c$, computed using Eq. (4.2) for a specific value of $\kappa_\lambda(\theta_E)$. In this calculation, $\alpha(\theta_E)$ entering Eq. (4.2) is the deviation angle of the full core-MSD model, *computed at a fixed angle corresponding to the peak posterior value of θ_E found in the pure PL MCMC run of Fig. 5.5*. In green, we show the value given by Eq. (5.11). We see that Eq. (5.11) leads to a bound on θ_c which is a factor of 2 or so stronger (that is, more conservative) than the MCMC bound.

5.4 Simulations

The results of Ref. [BT21b], reported in Sec. 5.1, 5.2, 5.3, suggest the possibility to use time delay cosmography, together with a H_0 prior, to detect or constrain ULDM cores presence in massive ellipticals. In fact, if ULDM cores may cause an H_0 shift as described in Fig. 5.2, then the same shift can be converted as a signal for the presence of a core, if a prior on H_0 is used. This is apparent in Fig. 5.6, where a pipeline with an H_0 prior manages to see a signal of the soliton presence in the mock system. We can think of applying the same principle to actual imaging data. We call this method of searching for ultralight axions via time-delay cosmography AxionH0graphy.

However, estimates on how much mass can actually relax in a solitonic core, based on Eq. (5.23), are rather simplistic. For sharper theoretical predictions of the AxionH0graphy effect, on [BT25] we here extended the analysis of Ref [BT21b] by using numerical simulations to study ULDM dynamics on the relevant galactic scales. The following is based on Ref. [BT25]. We focus on $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$, $\alpha_\chi \approx 0.15$ consistent with all other observational constraints, and TDCOSMO-like galaxies [Mil+20]: massive ellipticals with $M_{200} \approx 10^{13} M_\odot$ and lens and source redshift configurations typical in strong lensing.

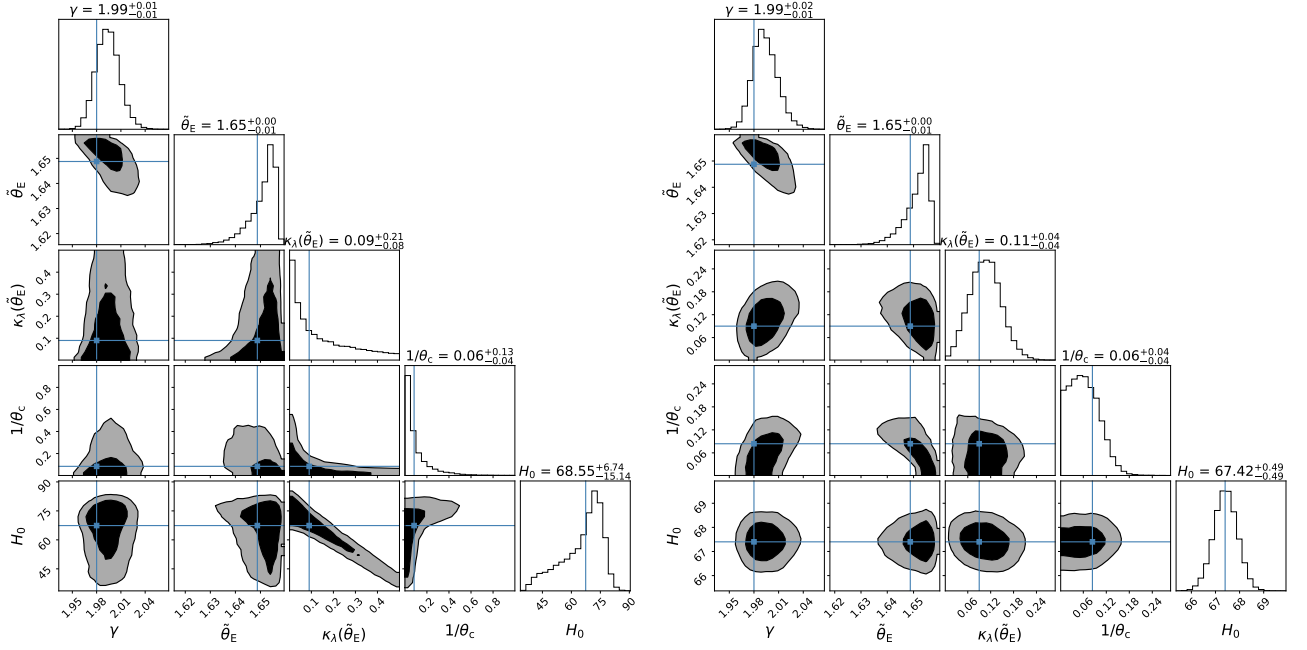


FIGURE 5.6: Lensing analysis for the same mock data as in Fig. 5.5, this time adding a core component to the fit. The blue lines correspond to the true values used for the mock. **Top:** Model inference with flat prior on H_0 . We remark that the mock is consistent with a no-core solution; hence, the median value showed for $1/\theta_c$ is an artefact of the finite range of the prior. The MSD is manifest by the broadening of the H_0 posterior distribution. Code: $\odot$. **Bottom:** Same as in the top panel, this time adding a CMB prior on H_0 . As expected, the no-core solution is now disfavoured. Code: $\odot$.

5.4.1 Simulation setup

We consider $10^{-25} \leq m/\text{eV} \leq 10^{-24}$, and simulate galaxy halos with halo mass M_{200}^{halo} varying around $10^{13} M_\odot$, and cosmological ULDM fraction $M_{\text{ULDM}} \approx \alpha_\chi M_{200}^{\text{halo}}$, fixed to $\alpha_\chi = 0.15$ in most of the analysis.

We use a pseudo-spectral Schrödinger-Poisson solver described in App. A.1, based on the method of Ref. [LPT18], and implemented on a discretized grid with box length $L \sim 1 \text{ Mpc}$ varying slightly between different simulations. This box length is chosen as compromise between our computational resources and the need to resolve the wave dynamics of ULDM in $M_{200}^{\text{halo}} \approx 10^{13} M_\odot$ halos, which have R_{200} of a few hundred kpc.

Since ULDM makes-up only a subdominant fraction of the total DM, we need to implement also the dominant CDM component, as well as the stellar mass component in the halo (in TDCOSMO-like lens galaxies, the stellar mass is comparable to the DM mass near the projected Einstein radius). Our simulations cannot yet handle cosmological initial conditions, nor approximate realistic stellar physics. We therefore explore the sensitivity of the results to different approximate prescriptions for treating the galaxy halo:

1. *CDM and stars as static external potential:* Our main results are obtained in a simple set-up, in which we approximate the dominant CDM and stellar mass distributions of the lens galaxy by fixed static external potentials. In this case, the numerical simulation includes only the ULDM dynamics, subject to the influence of the external potential.

We approximate the CDM component, with mass $M^{\text{CDM}} = (1 - \alpha_\chi) M_{200}^{\text{halo}}$, as an NFW potential. The NFW scale radius parameter R_s in simulations is taken from the mass-concentration relations of [MDB08]. The stellar component is taken to follow a Hernquist potential [Her90].

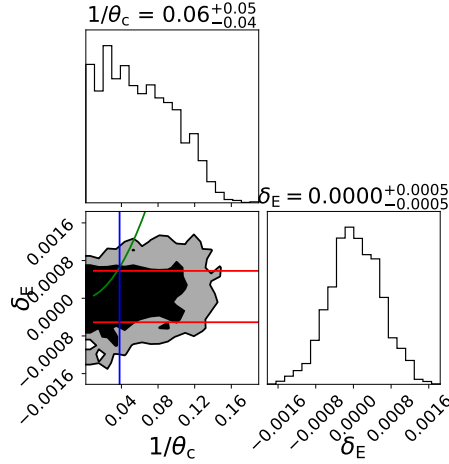


FIGURE 5.7: Triangle plot of δ_E and $1/\theta_c$, calculated for a fixed $\kappa_\lambda(\tilde{\theta}_E)$. In green, the δ_E coming from Eq. (5.11). Notice that the maximum δ_E allowed for our mock is $|\delta_E| \lesssim 0.0008$ (horizontal red lines). Combining this with Eq. (5.11) would yield the naive bound $\theta_c \gtrsim 25''$ (blue vertical line, marking the intersection of the green and red lines). However, the region explored by the MCMC suggests that the more realistic bound is somewhat weaker, $\theta_c \gtrsim 10''$. Code: [Q](#).

2. *CDM as a heavier field with $m_{\text{heavy}} \gg m$* : Here we approximately implement CDM by adding to the code another dynamical ULDM field, with a larger particle mass $m_{\text{heavy}} \gg m$; for simplicity, we continue to refer to this component as “CDM”. The two fields interact with each other only via gravity. The approximation of CDM by a large- m Schrödinger field relies on the Vlasov-Poisson correspondence [Moc+18]. We maintain sufficient numerical resolution to capture the de Broglie wavelength of the more massive field [LHB19; MS21]. The highest mass we can afford to simulate is $m_{\text{heavy}} = 10^{-23}$ eV. Hence the ratio between m_{heavy} and the lighter “proper ULDM” m in our analysis varies between $10 \leq m_{\text{heavy}}/m \leq 100$. In these simulations, the stellar component is neglected. Thus these simulations cannot be used directly to study gravitational lensing results. The goal of this analysis is to test the effect of backreaction of the subdominant ULDM field on the dominant CDM.

Simulating the background halo CDM with a dynamical field allows us to also test the impact of different initial conditions, notably the effect of galaxy mergers.

In all simulations, the ULDM density distribution is initialized approximately as an α_χ -scaled copy of the dominant CDM density profile. Initial conditions are defined via the Eddington procedure [WK93; Lan+20], adapted to wave equations:

$$\psi_i(x) = (\Delta v)^{3/2} \sum_{\vec{v}} \sqrt{f(\mathcal{E}(x, v))} e^{im_i \vec{x} \cdot \vec{v} + i\varphi_{\vec{v}}}, \quad (5.25)$$

where ψ_i is the Schrödinger field associated to m_i (the index i refers to the proper ULDM field m and also to the m_{heavy} implementation of “CDM”, where relevant), $\varphi_{\vec{v}}$ is the random phase associated with the Fourier component of velocity $\vec{v} = \vec{k}/m_i$, Δv is the velocity resolution step available in the simulation, and

$$f(\mathcal{E}) = \frac{2}{\sqrt{8}\pi^2} \int_0^{\sqrt{\mathcal{E}}} dQ \frac{d^2 \rho}{d\Psi^2}(Q), \quad (5.26)$$

$$\mathcal{E} = \Psi(r) - \frac{v^2}{2}, \quad \Psi = -\Phi + \Phi(r_{\text{max}}), \quad (5.27)$$

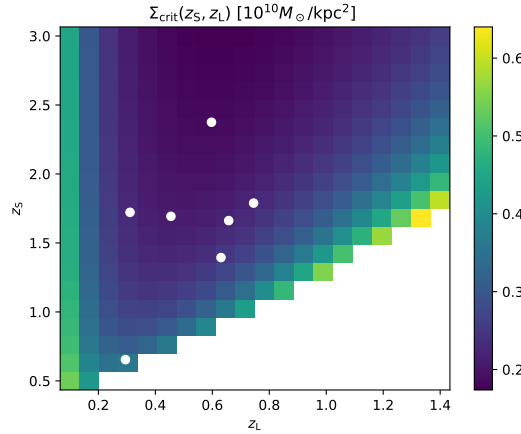


FIGURE 5.8: $\Sigma_{\text{cr}}(z_l, z_s)$ in the (z_l, z_s) plane. TDCOSMO systems [Mil+20] are marked by dots.

where ρ is the mass density of the simulated field, Φ is the total gravitational potential and $Q = \sqrt{\mathcal{E} - \Psi}$. Further details on the implementation of this initial condition are in App. A.6. The random phases $\phi_{\vec{v}}$ cause interference patterns that make runs with different random seeds differ from each other on scales smaller than λ_{dB} , even if they are designed to give the same density profile on average.

Unless stated otherwise, simulations begin at $t = 0$ and final results are read out at the Hubble time $t = t_H = 1/H_0$.

5.4.2 Simulation results

Our simulations produce results in terms of mass density profiles. However, gravitational lensing depends not only on the intrinsic density profile of the lens galaxy, but also on projection effects related to the observer-lens-source configuration. In particular, mapping mass column density into lensing convergence requires the critical density $\Sigma_{\text{cr}} = D_s / (4\pi G D_{\text{ls}} D_l)$, see Eq. (2.29). Fig. 5.8 shows Σ_{cr} in the (z_l, z_s) plane (with the constraint $z_s > z_l + 0.3$). The seven lens systems from TDCOSMO [Mil+20] are marked by white dots. Their Σ_{cr} ranges from 1.8 to 3.4, with a mean of $2.4 \times 10^9 M_\odot / \text{kpc}^2$. In presenting results that require convergence, we either use a reference value of $\Sigma_{\text{cr}} = 2 \times 10^9 M_\odot / \text{kpc}^2$, or refer to specific examples from the TDCOSMO sample.

Static external CDM and stars

We begin with simulations in which the dominant CDM and stellar components of the lens galaxy are approximated using static external NFW+Hernquist potentials. The dynamical ULDM has a cosmological fraction of $\alpha_\chi = 0.15$. The CDM NFW profile has $M_{200}^{\text{halo}} \approx 10^{13} M_\odot$ with scale radius $R_s \approx 70$ kpc. The stellar Hernquist profile has a total mass of $\approx 7.5 \times 10^{11} M_\odot$ and scale radius ≈ 7 kpc. At $r = 10$ kpc the enclosed stellar mass is roughly twice the enclosed DM mass, compatible with estimates for typical TDCOSMO systems.

Fig. 5.9 shows the radially-averaged 3D mass density for $m = 2 \times 10^{-25}$ eV. We show H_0 signal δ_H (top number in blue) and imaging error δ_E (bottom number in red) for three redshift configurations (z_s, z_l) , corresponding to TDCOSMO systems⁴. Each redshift configuration maps to different projected Einstein radius $D_l \theta_E$, indicated on the plot by thin vertical lines connected to the corresponding δ_H and δ_E . Colormap inset shows ULDM convergence κ_c , using a typical reference value of $\Sigma_{\text{cr}} = 2 \times$

⁴From left to right in Fig. 5.9, these systems are RXJ1131, PG1115, and DESJ0408. Other TDCOSMO systems have (z_l, z_s) combinations that place them in between these three in Fig. 5.9.

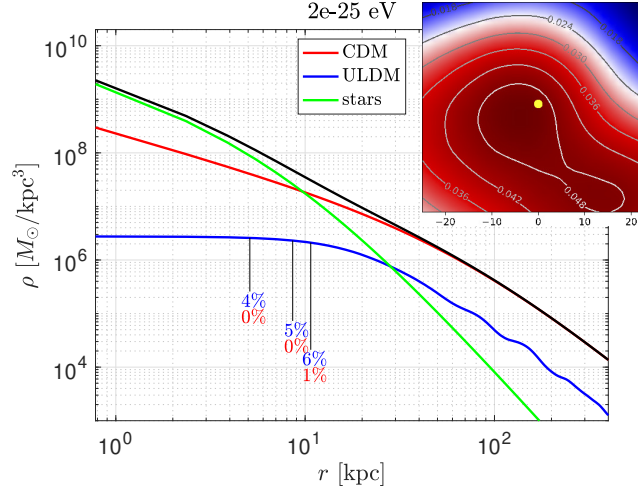


FIGURE 5.9: Radially-averaged density. Here $m = 2 \times 10^{-25}$ eV, $\alpha_\chi = 0.15$. Background halo implemented as static NFW+Hernquist potentials. We show H_0 signal δ_H (top number in blue) and imaging error δ_E (bottom number in red) for three redshift configurations (z_1, z_s) , corresponding to real TDCOSMO systems. Each redshift configuration maps to slightly different projected Einstein radius $D_1 \theta_E$. Colormap inset shows ULDM convergence κ_c using $\Sigma_{\text{cr}} = 2 \times 10^9 \text{ M}_\odot/\text{kpc}^2$ (grid units: kpc). Yellow spot marks the lens halo center of mass.

$10^9 \text{ M}_\odot/\text{kpc}^2$ to translate column density to lensing convergence (grid units: kpc). Yellow spot marks the halo center of mass.

Fig. 5.9 showcases the AxionH0graphy scenario. ULDM is predicted to produce an MSD with $\kappa_c(\theta_E)$ of a few percent, that would be essentially invisible in imaging and very challenging to detect via stellar kinematics alone. Since the ULDM core falls off and joins the dominant CDM halo morphology after a few tens of kpc, it would also be difficult to detect in weak lensing, while cosmological constraints allow $\alpha_\chi = 0.15$. Thus, if cosmography delivers $\lesssim 2\%$ precision in inferred H_0 [BT21a] when the MSD is *not* modelled explicitly, then Fig. 5.9 demonstrates a physically-motivated scenario in which an ultralight axion may be first discovered from an apparent H_0 tension between the strong gravitational lensing measurement and other probes, e.g. early universe ones.

Fig. 5.10 extends the results to $m = (1, 2, 5, 10) \times 10^{-25}$ eV (this contains again the result of Fig. 5.9 for comparison).

For $m = 10^{-25}$ eV (top-left panel) the core extends far beyond the Einstein radius, making the MSD very precise. The ULDM-induced convergence is small – about a factor of 5 smaller than α_χ – causing a small H_0 signal $\delta_H \lesssim 3\%$ that is still, however, somewhat larger than cosmography precision expected in forecasts [BT21a]. It is interesting to also investigate the regime $m < 10^{-25}$ eV, but our simulations do not extend there because of our technical limitation in simulating a sufficiently large box.

$m \approx 2 \times 10^{-25}$ eV (top-right panel, or Fig. 5.9) is a “sweet-spot” for AxionH0graphy with H_0 signal $\delta_H \sim 10\%$.

For $m = 5 \times 10^{-25}$ eV (bottom-left) the spatial scale of interference patterns in the ULDM field, the displacement of the ULDM core from the center of mass of the dominant CDM+stars halo [LHB19; LHY21]; and the core radius itself, which are all of the order of λ_{dB} , approach the scale of the projected Einstein radius for a TDCOSMO-like galaxy. This causes deviations from the MSD approximation, seen in the bottom-left panel of Fig. (5.10) with $\delta_E \gtrsim 4\%$. In this regime the spherical approximation used in Eq. (4.3) becomes less accurate. Mock data analysis is needed to investigate this regime quantitatively. As a tentative rough estimate of the impact on imaging, noting that Eq. (4.3) is the spherical average of

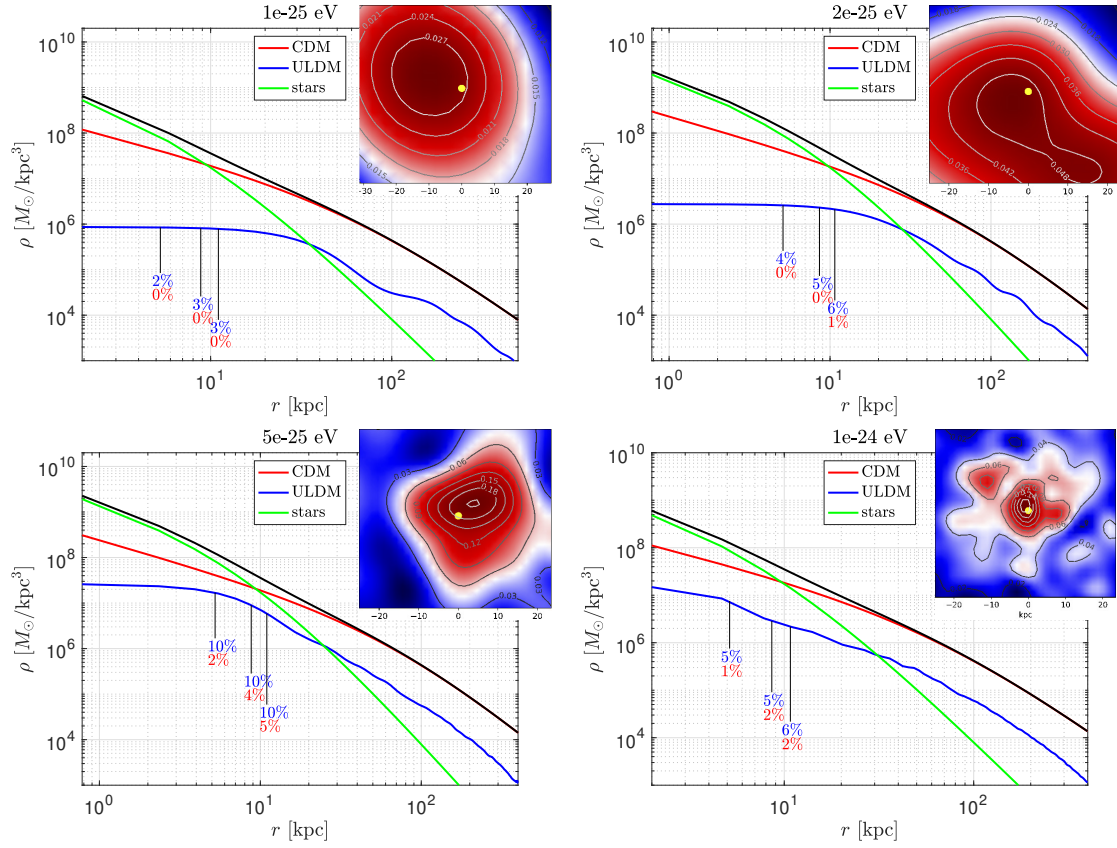


FIGURE 5.10: Same set-up as Fig. 5.9 (reproduced in the top-right panel), extended to $m = (1, 5, 10) \times 10^{-25}$ eV (top-left, bottom-left, and bottom-right panels, respectively).

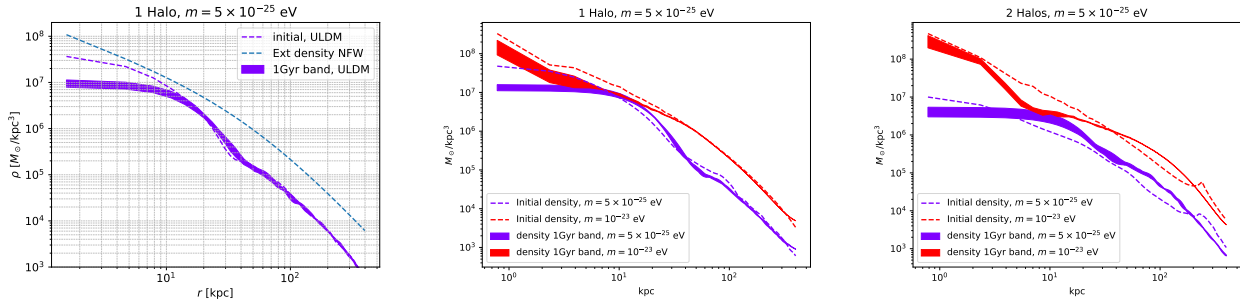


FIGURE 5.11: Simulation results for $m = 5 \times 10^{-25}$ eV, $\alpha_\chi = 0.15$. Panels differ in the implementation of the CDM halo component, and in initial conditions. Left panel: CDM implemented as static NFW potential. Middle panel: CDM implemented by another dynamical field with $m_{\text{heavy}} = 10^{-23}$ eV. Right panel: CDM implemented as $m_{\text{heavy}} = 10^{-23}$ eV, initialized as two halos that merge dynamically. Final time results are shown by thick bands that contain the extreme values of the density field during the time $t = t_H \pm 1$ Gyr, capturing the effect of core oscillations. Initial conditions are shown by dashed lines.

the difference between $\kappa_c(\vec{\theta})$ and its value at θ_E , we can compare δ_E to the maximal variation in $\kappa_c(\vec{\theta})$ with respect to $\kappa_c(\theta_E)$, taken across the region $|\vec{\theta}| \lesssim \theta_E$:

$$\Delta\kappa_c := \max_{|\vec{\theta}| < \theta_E} (|\kappa_c(\vec{\theta}) - \kappa_c(\theta_E)|), \quad (5.28)$$

where the coordinates are centered on the CDM halo center of mass. Choosing $D_1\theta_E \approx 10$ kpc for the boundary of the region, and using $\Sigma_{\text{cr}} = 2 \times 10^9 \text{ M}_\odot/\text{kpc}$ for reference, we find $\Delta\kappa_c \approx 0.09$ for the $m = 5 \times 10^{-25}$ eV simulation. Thus ULDM interference patterns and core displacement cause imaging errors of comparable magnitude to those caused by the finite-core effect.

For $m = 10^{-24}$ eV (bottom-right), the core radius is smaller than $D_1\theta_E$ and the convergence map shows that the MSD approximation is poor, violating Eq. (4.3) at the $\mathcal{O}(1)$ level.

Dynamical CDM via $m_{\text{heavy}} \gg m$

The middle panel of Fig. 5.11 shows simulation results for $m = 5 \times 10^{-25}$ eV, where the CDM component of the halo is simulated by a dynamical field with $m_{\text{heavy}} = 10^{-23}$ eV. Initial conditions are shown by dashed lines. Final time results are shown by thick bands that contain the extreme values of the density field during the time $t = t_H \pm 1$ Gyr. The thickness of these bands demonstrates the amplitude of core oscillations and dynamical interference patterns. (Although we did not highlight this fact, comparable core oscillations exist in the simulations of Fig. 5.10.)

For reference, the left panel of Fig. 5.11 shows results obtained with a fixed static NFW background, which approximately matches the CDM initial conditions in the dynamical case. These simulations do not include a stellar mass component: the goal is to investigate the impact of the dynamical background on the ULDM core in a simple toy model.

To test the sensitivity of our results to initial conditions, notably major mergers, we also perform runs initialized as two NFW halos of similar initial mass profiles that merge dynamically. The two halos are initially separated by ≈ 100 kpc, at rest w.r.t. each other. The result is shown in the right panel of Fig. 5.11.

Fig. 5.11 suggests that ULDM cores are not very sensitive to the details of CDM backreaction or halo initial conditions. In all three panels, the ULDM core central density is similar to within about a factor of two, with a similar core radius of about 20 kpc. Note that our simulations can only roughly resolve

$\frac{m}{10^{-25}\text{eV}}$	$\frac{M_{200}}{10^{12}M_\odot}$	$\frac{\rho_{\text{sol}}(0)}{10^3 M_\odot/\text{pc}^3}$	$\frac{M_{\text{sol}}}{10^{11}M_\odot}$	$\frac{r_c}{\text{kpc}}$	$\frac{\lambda_{\text{dB}}}{2\text{kpc}}$	$\frac{r_E}{\text{kpc}}$	$10^2\delta_E$	$10^2\Delta\kappa_c$	$\delta_H \approx \kappa_c(\theta_E)$	Conditions
1	13.6	$0.4^{+0.1}_{-0.0}$	$2.6^{+1.1}_{-0.0}$	37^{+8}_{-0}	45	$8.4^{+3.1}_{-3.1}$	$0.6^{+0.1}_{-0.2}$	$0.3^{+0.1}_{-0.2}$	$0.01^{+0.00}_{-0.01}$	1 halo, ext NFW
1	13.6	$0.9^{+0.3}_{-0.2}$	$3.4^{+1.1}_{-0.0}$	33^{+4}_{-0}	45	$8.4^{+3.1}_{-3.1}$	$1.5^{+0.2}_{-0.6}$	$0.4^{+0.3}_{-0.3}$	$0.03^{+0.00}_{-0.01}$	1 halo, ext NFW + Hernq.
1	11.1	$0.4^{+0.2}_{-0.1}$	$5.3^{+6.6}_{-0.0}$	48^{+12}_{-2}	48	$4.4^{+2.2}_{-2.2}$	$0.8^{+0.3}_{-0.2}$	$0.1^{+0.1}_{-0.0}$	$0.01^{+0.00}_{-0.00}$	1 halo, 2 fields
1	45.6	$6.2^{+4.8}_{-2.0}$	$33.2^{+0.0}_{-17.0}$	27^{+9}_{-2}	30	$26.7^{+9.0}_{-9.0}$	$1.9^{+1.3}_{-0.2}$	$7.6^{+6.3}_{-8.0}$	$0.13^{+0.05}_{-0.08}$	1 halo, 2 fields
1	5.3	$0.2^{+0.0}_{-0.1}$	$4.0^{+0.0}_{-1.6}$	56^{+23}_{-0}	62	$4.8^{+1.8}_{-1.8}$	$0.4^{+0.2}_{-0.1}$	$0.0^{+0.0}_{-0.0}$	$0.01^{+0.00}_{-0.00}$	2 halos, 2 fields
1	38.0	$4.8^{+7.5}_{-1.0}$	$38.2^{+7.4}_{-6.7}$	38^{+2}_{-10}	32	$18.3^{+4.7}_{-4.7}$	$2.6^{+1.0}_{-0.6}$	$5.1^{+6.3}_{-4.5}$	$0.21^{+0.03}_{-0.10}$	3 halos, 2 fields
2	21.0	$15.2^{+4.4}_{-6.7}$	$7.7^{+3.2}_{-0.0}$	17^{+3}_{-0}	20	$15.7^{+5.0}_{-5.0}$	$2.0^{+0.4}_{-0.7}$	$5.5^{+2.6}_{-3.4}$	$0.12^{+0.02}_{-0.05}$	1 halo, ext NFW
2	12.8	$2.7^{+1.9}_{-0.0}$	$5.9^{+0.0}_{-2.1}$	20^{+2}_{-0}	23	$7.9^{+2.9}_{-2.9}$	$0.8^{+0.5}_{-0.1}$	$0.9^{+1.2}_{-0.7}$	$0.05^{+0.01}_{-0.02}$	1 halo, ext NFW + Hernq.
2	7.8	$0.8^{+0.8}_{-0.0}$	$6.5^{+0.0}_{-2.2}$	36^{+0}_{-7}	27	$3.5^{+1.7}_{-1.7}$	$3.0^{+2.5}_{-0.1}$	$0.2^{+0.5}_{-0.2}$	$0.05^{+0.00}_{-0.02}$	1 halo, 2 fields
2	24.0	$14.5^{+4.5}_{-2.3}$	$14.7^{+0.0}_{-0.7}$	22^{+0}_{-2}	19	$15.4^{+5.4}_{-5.4}$	$3.3^{+1.3}_{-0.8}$	$12.0^{+10.1}_{-10.4}$	$0.24^{+0.07}_{-0.13}$	1 halo, 2 fields
2	21.0	$9.3^{+8.4}_{-5.5}$	$10.7^{+10.3}_{-1.4}$	23^{+3}_{-5}	20	$11.2^{+3.8}_{-3.8}$	$2.6^{+1.2}_{-0.5}$	$7.0^{+8.9}_{-6.2}$	$0.20^{+0.04}_{-0.10}$	2 halos, 2 fields
2	11.3	$1.5^{+4.6}_{-0.0}$	$14.6^{+0.0}_{-6.0}$	34^{+0}_{-9}	24	$7.7^{+2.1}_{-2.1}$	$1.3^{+0.9}_{-0.1}$	$1.5^{+0.9}_{-1.0}$	$0.07^{+0.03}_{-0.03}$	3 halos, 2 fields
5	5.4	$7.9^{+3.4}_{-0.1}$	$3.0^{+0.0}_{-0.8}$	14^{+3}_{-0}	12	$4.2^{+1.6}_{-1.6}$	$10.9^{+2.9}_{-3.5}$	$1.9^{+3.7}_{-1.9}$	$0.13^{+0.01}_{-0.06}$	1 halo, ext NFW
5	13.4	$25.9^{+19.1}_{-7.3}$	$2.3^{+0.0}_{-0.6}$	7^{+3}_{-0}	9	$8.4^{+3.0}_{-3.0}$	$0.3^{+0.3}_{-0.0}$	$9.1^{+7.8}_{-7.8}$	$0.15^{+0.05}_{-0.08}$	1 halo, ext NFW + Hernq.
5	5.5	$15.9^{+0.0}_{-5.9}$	$3.2^{+0.4}_{-0.0}$	12^{+0}_{-0}	12	$4.7^{+1.8}_{-1.8}$	$6.4^{+6.3}_{-0.4}$	$4.0^{+7.0}_{-3.5}$	$0.18^{+0.03}_{-0.08}$	1 halo, 2 fields
5	5.2	$1.1^{+2.3}_{-0.1}$	$1.9^{+0.1}_{-1.0}$	18^{+0}_{-5}	12	$4.5^{+1.8}_{-1.8}$	$1.1^{+1.3}_{-0.2}$	$0.7^{+1.1}_{-0.6}$	$0.03^{+0.01}_{-0.02}$	1 halo, 2 fields
5	7.6	$3.3^{+8.4}_{-0.3}$	$1.2^{+1.6}_{-0.0}$	15^{+2}_{-0}	11	$6.1^{+2.1}_{-2.1}$	$1.1^{+0.9}_{-0.0}$	$2.6^{+2.3}_{-2.0}$	$0.06^{+0.01}_{-0.03}$	2 halos, 2 fields
5	7.7	$7.4^{+0.8}_{-0.0}$	$2.4^{+0.1}_{-0.1}$	15^{+0}_{-2}	11	$6.3^{+1.6}_{-1.6}$	$2.5^{+2.6}_{-0.2}$	$2.1^{+3.2}_{-2.0}$	$0.10^{+0.02}_{-0.05}$	3 halos, 2 fields
10	9.9	$8.0^{+1.4}_{-0.0}$	$0.4^{+0.0}_{-0.1}$	8^{+0}_{-0}	5	$6.8^{+2.5}_{-2.5}$	$1.9^{+1.8}_{-0.1}$	$7.1^{+4.7}_{-5.6}$	$0.10^{+0.04}_{-0.06}$	1 halo, ext NFW
10	12.8	$17.4^{+4.1}_{-1.5}$	$0.2^{+0.0}_{-0.0}$	5^{+0}_{-0}	5	$8.0^{+2.9}_{-2.9}$	$2.4^{+1.3}_{-0.4}$	$3.1^{+2.4}_{-2.9}$	$0.06^{+0.02}_{-0.03}$	1 halo, ext NFW + Hernq.
10	7.6	$16.1^{+7.8}_{-6.0}$	$1.0^{+1.1}_{-0.1}$	9^{+0}_{-2}	5	$5.7^{+2.3}_{-2.3}$	$0.2^{+0.1}_{-0.0}$	$4.5^{+3.5}_{-3.3}$	$0.10^{+0.02}_{-0.05}$	1 halo, 2 fields
10	8.4	$35.9^{+0.0}_{-11.4}$	$2.1^{+0.8}_{-0.0}$	9^{+0}_{-0}	5	$6.4^{+2.3}_{-2.3}$	$3.8^{+1.9}_{-0.7}$	$11.0^{+12.8}_{-10.0}$	$0.22^{+0.07}_{-0.12}$	2 halos, 2 fields
10	12.8	$17.2^{+0.0}_{-6.5}$	$0.8^{+1.3}_{-0.0}$	8^{+0}_{-0}	5	$9.5^{+2.5}_{-2.5}$	$0.9^{+0.3}_{-0.3}$	$4.4^{+1.4}_{-2.2}$	$0.08^{+0.01}_{-0.04}$	3 halos, 2 fields

TABLE 5.2: Summary of simulation results for $\alpha_\chi = 0.15$. From left to right: (1) ULDM particle mass, (2) M_{200} (final snapshot), (3) Central ULDM density (4) soliton mass from Eq. (5.29), (5) core radius, vis $\rho_{\text{ULDM}}(r_c) = 0.5\rho_{\text{ULDM}}(0)$, (6) half de-Broglie wavelength, using Eq. (5.31), (7) projected Einstein radius $r_E = D_1\theta_E$, (8) δ_E from Eq. (4.3) (in %), (9) maximal convergence variation from Eq. (5.28) (in %), (10) H_0 signal, or expected $\kappa_c(\theta_E)$, (11) simulated configuration.

the detailed dynamics of the $m_{\text{heavy}} = 10^{-23}$ eV “CDM” component, which exhibits interference patterns with typical wavelength of order 1 kpc. In particular, the right panel in Fig. 5.11 shows that the m_{heavy} field is forming a soliton of its own in the inner few kpc region of the halo. This could be interesting as it is expected that the lighter ULDM species, despite being subdominant in the total mass budget, could accelerate dynamical relaxation in the dominant CDM component [BBS22]. For AxionH0graphy, however, it is sufficient to notice that the ULDM core is not strongly dependent on the details of the main CDM component.

Results summary for $\alpha_\chi = 0.15$

Tab. 5.2 summarizes the main results of our simulations for $\alpha_\chi = 0.15$. For each of $m = (1, 2, 5, 10) \times 10^{-25}$ eV, we simulate a set of scenarios: single halo evolution with static CDM NFW potential; single halo with static CDM+stars; single halo with dynamical CDM implemented via $m_{\text{heavy}} = 10^{-23}$ eV⁵; two-halo merger with dynamical m_{heavy} CDM; and three-halo merger with dynamical m_{heavy} CDM. All simulations are run until $t = t_H$. For each simulation we quote the CDM halo mass M_{200} (extracted at $t = t_H$ when CDM is dynamical). We now discuss each additional column of Tab. 5.2 in turn.

⁵For $m = (1, 2, 5) \times 10^{-25}$ eV we report results of two separate runs.

Soliton mass: In Sec 5.1 [BT21b], we parametrized the ULDM soliton profile in the presence of external potential with the form (see Eq. (5.4))

$$\rho_{\text{sol}}(r) = \frac{\lambda_s^4}{(1 + a^2 \lambda^2 r^2)^{2b}} \frac{m^2}{4\pi G} , \quad (5.29)$$

For simplicity, we use this parameterization even though the field configurations we find in the simulations deviate from spherical symmetry. Fitting the core with Eq. (5.29), we find the soliton mass up to R_{200} as

$$M_{\text{sol}} = 4\pi \int_0^{R_{200}} dr r^2 \rho_{\text{sol}}(r) . \quad (5.30)$$

The error bars we quote for M_{sol} and $\rho_{\text{sol}}(0)$ come from maximum and minimum value in the period $t_H \pm 1$ Gyr.

Core scale radius r_c , and λ_{dB} : We denote the scale length of the core in simulations by r_c , defined as the radius at which the ULDM density decreases by a factor of 2 from its peak value. r_c is comparable with $\lambda_{\text{dB}}/2$, estimated by

$$\lambda_{\text{dB}} \approx 20 \text{ kpc} \left(\frac{180 \text{ km s}^{-1}}{v} \right) \left(\frac{5 \times 10^{-25} \text{ eV}}{m} \right) , \quad (5.31)$$

where v is the 1D velocity dispersion

$$v \approx \sqrt{\frac{GM_{200}}{3R_{200}}} \approx 180 \text{ km s}^{-1} \left(\frac{M_{200}}{10^{13} M_{\odot}} \right)^{1/3} , \quad (5.32)$$

using $R_{200} \approx 450 \text{ kpc} \left(\frac{M_{200}}{10^{13} M_{\odot}} \right)^{1/3}$.⁶

Again, the error bars we quote for r_c come from maximum and minimum value in the period $t_H \pm 1$ Gyr.

Einstein radius r_E : It is useful to compute the projected Einstein radius $r_E = D_1 \theta_E$ for comparison with r_c . We expect precise MSD when $r_E \ll r_c$. In simulations that do not include a stellar mass component, we assign a value for θ_E by adding to the convergence computation a contribution from an Hernquist profile, scaling the stellar mass linearly with M_{200} . The error bars we quote for r_E are intended to represent projection effects. They relate to the smallest and largest θ_E values obtained across the seven TDCOSMO systems (see Fig. 5.8, and compare also to the vertical thin lines in Figs. 5.9 and 5.10).

Imaging error estimates δ_E and $\Delta\kappa_c$: We quote both δ_E (Eq. (4.3)) and $\Delta\kappa_c$ (Eq. (5.28)) as estimates of imaging errors. Error bars refer to maximum and minimum values obtained from the redshift configurations of the TDCOSMO sample.

We highlight in **red** δ_E and $\Delta\kappa_c$ entries that exceed 0.03, as these halos may exhibit detectable MSD-breaking imaging errors. Note again that the error estimates do not take into account degeneracies involving external shear and flexion [Teo+22; TB23].

H_0 signal: We use Eq. (4.3) to estimate H_0 signal, or equivalently the expected $\kappa_c(\theta_E)$, from the radial-averaged convergence profile. The error bars we show come from the maximum and minimum values of $\kappa_c(\theta_E)$ obtained by the redshift configurations of TDCOSMO systems.

Finally, it is important to note that the ULDM mass in our simulated halos is matched to the cosmological fraction α_χ . If cosmological evolution causes significant suppression of the ULDM fraction on distance scales of $\sim \text{Mpc}$, the actual ULDM mass in the inner region of galaxies may be lower than α_χ . Our

⁶In estimating λ_{dB} , which is only linearly sensitive to the value of v , and moreover only referred to in rough parametric estimates, we use the mean value of v in the virial region of the halo. Later when we discuss dynamical relaxation, we would need to be a bit more precise and to estimate v self-consistently in the region $r \sim \lambda_{\text{dB}}$.

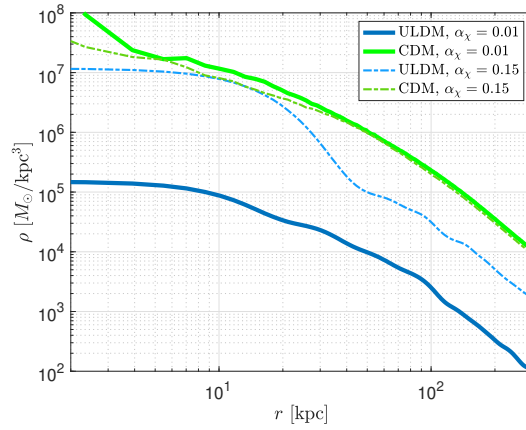


FIGURE 5.12: Comparison of results for $\alpha_\chi = 0.01$ (thick solid lines) and $\alpha_\chi = 0.15$ (thin dot-dashed lines), done for $m = 5 \times 10^{-25}$ eV, with dynamical CDM modelled via m_{heavy} field.

results could then exaggerate the detectability prospects of subdominant ULDM. The results in this work must be taken as approximations, which need to be validated via more refined simulations. The AxionH0graphy scenario may be misguided, if the approximations used to obtain our results prove to significantly overestimate the expected amount of κ_c in lens galaxies. This adds motivation to extend the analysis to full cosmological simulations which include N -body interactions for CDM and stars in future work.

5.4.3 Brief view of $\alpha_\chi < 0.15$

Our work focused on $\alpha_\chi = 0.15$, which is roughly the maximal ULDM fraction consistent with cosmological data [Kob+17; Lag+22]. Here we take a brief glance at smaller α_χ .

Fig. 5.12 compares simulations for $\alpha_\chi = 0.01$ (thick solid lines) and $\alpha_\chi = 0.15$ (thin dot-dashed lines), done for $m = 5 \times 10^{-25}$ eV. Results for $\alpha_\chi = 0.15$ are the same as in the middle panel of Fig. 5.11. The comparison highlights the soliton core that begins to form in the $\alpha_\chi = 0.15$ case, but not in the $\alpha_\chi = 0.01$ case, in qualitative agreement with results in [Lag+24; Sch+20].

5.4.4 Timescale of core formation

So far we focused on simulation snapshots at $t = t_H$. We now comment on the time evolution leading to these snapshots.

Fig. 5.13 shows the time evolution of the peak ULDM density of two simulations, containing $m = 2 \times 10^{-25}$ eV (left panel) and 5×10^{-25} eV (right panel) at $\alpha_\chi = 0.15$ with dynamical m_{heavy} CDM. We can identify three regimes in the dynamics of the core:

1. *Wave pressure expels mass outwards, forming flattened region.* As already explained, in the (simplistic) initial conditions of our simulations the ULDM halo is an approximate copy of the dominant CDM NFW halo, scaled-down by a factor of α_χ . The Eddington prescription used to initialize the field does not take into account wave mechanics pressure. This omission is analogous to neglecting two-body stellar encounters when assigning initial conditions in standard N -body simulations. Once we start the run, ULDM in the dense central region of the halo evolves quickly due to wave

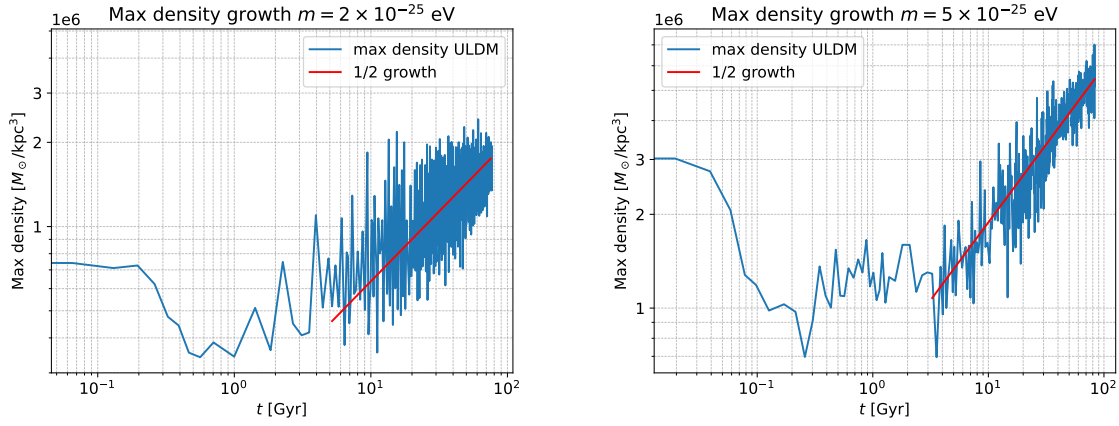


FIGURE 5.13: ULDM core density as a function of time. Left (right) panel: $m = 2$ (5) $\times 10^{-25}$ eV.

pressure to form a flattened core, on a wave-mechanics time scale t_{dB} given by

$$t_{\text{dB}} = \frac{1}{mv^2} \approx 0.1 \left(\frac{200 \text{ km/s}}{v} \right)^2 \left(\frac{5 \times 10^{-25} \text{ eV}}{m} \right) \text{ Gyr}. \quad (5.33)$$

t_{dB} is comparable but probably somewhat shorter than the (m -independent) free-fall time,

$$t_{\text{freefall}} = \sqrt{\frac{3\pi R_{200}^3}{32GM_{200}}} \approx 0.8 \text{ Gyr}. \quad (5.34)$$

We do not investigate the details of the initial short-time dynamics, because these necessarily depend on the details of our rather ad-hoc initial conditions. The only comment we make in this regard is that the later evolution of the core is relatively independent on the details of the initial conditions, as those are quickly “ironed out” by the wave pressure.

2. *Dynamical relaxation and emergence of a soliton.* After the initial density drop, the core density fluctuates for a while without substantial mean growth.

The state of the field during this stage may not be very different from the random noise state used in Ref. [LPT18; BOFT19; BOFT21] to investigate soliton condensation in a statistically homogeneous background. The relaxation timescale t_{rel} in that case was found to be Eq. (5.19), which we rewrite as

$$t_{\text{rel}} \approx \frac{\sqrt{2}}{12\pi^3} \frac{m^3 \sigma^6}{G^2 \rho_\chi^2 \Lambda_C}, \quad (5.35)$$

where ρ_χ refers to the ULDM density. The simulations of soliton condensation from random noise, done in Ref. [LPT18], show self-gravitating solitons emerging from noise around $t \approx t_{\text{rel}}$, after which time the soliton mass grows approximately as $M_{\text{sol}} \propto t^{\frac{1}{2}}$.

To compare this expectation to our simulations, we can read ρ_χ directly from Fig. 5.13. However, t_{rel} depends very sensitively on the local velocity dispersion v in Eq. (5.35), which is not a straightforward quantity to define precisely in our situation, where dispersion sourced by the background potential adds to the motion of the ULDM core around the halo center of mass. We make a crude estimate as follows. From the simulation, we read the core radius r_c of the ULDM; we find $r_c \approx 26$ (18) kpc for $m = 2$ (5) $\times 10^{-25}$ eV, with enclosed background halo mass $M_{\text{halo}}(r_c) \approx 0.35$ (0.2) $\times 10^{12} M_\odot$. We then estimate the circular velocity at r_c as $v_{\text{circ}}^2(r_c) = GM_{\text{halo}}(r_c)/r_c$,

and use $\sigma \approx \sqrt{3} \times v_{\text{circ}}$ in Eq. (5.35). This crude estimate gives $v_{\text{circ}}(r_c) \approx 245$ (222) km/s, and $t_{\text{rel}} \approx 0.9$ (8) Gyr, for $m = 2$ (5) $\times 10^{-25}$ eV, corresponding to the left (right) panel of Fig. 5.13. This estimate is not very far from the behavior seen in Fig. 5.13. However, to appreciate how sensitive these estimates are to numerical details, note that using $r_c \rightarrow 2 \times r_c$ in our estimate would multiply the resulting t_{rel} by a factor of ~ 8 . We can only conclude that our results in Fig. 5.13 are not clearly inconsistent with Eq. (5.35).

3. *Soliton growth.* In the third stage of the evolution the central density in our simulations shows a clear trend of growth, consistent with $\rho_c \propto t^{\frac{1}{2}}$. To guide the eye, we superimpose a red line with this slope in both panels.

Note that we see $\rho_c \propto t^{\frac{1}{2}}$, while Ref. [LPT18] reported that the *mass* of the soliton, M_{sol} , grows as $\propto t^{\frac{1}{2}}$. This scaling is consistent if the soliton is not self-gravitating, but rather dominated by the velocity dispersion due to the background potential, in which case the soliton radius only weakly depends on the soliton mass and one finds $M_{\text{sol}} \propto \rho_c$. This is indeed the relevant regime in our simulations.

We track the build-up of the soliton to $t \gg t_H$. We do not see a clear break of the soliton growth, although we cannot exclude the beginning of a break at late time. Saturation of soliton growth is expected to occur when the velocity dispersion due to the soliton's self-gravity becomes comparable to the mean velocity dispersion in the halo (see discussion and references in [Bar+18; BBS22]). The solitons in our simulations do not reach this point at $t = t_H$; even the relatively prominent ULDM cores seen in the middle and right panels of Fig. 5.11, where the ULDM density is comparable to the CDM density near the core radius, still induce a contribution to v_{circ} that is smaller by a factor of few compared to the CDM-dominated peak halo v_{circ} . This result is consistent with expectations in Ref. [BT21b] that identified soliton growth as a bottleneck for AxionH0graphy.

5.5 Summary of the AxionH0graphy effect

In the context of the internal MSD as a possible bias for the lensing H_0 measurement, we explored ULDM as a concrete, well-motivated model that could naturally produce the required cores. We studied the lensing imprint of these cores and demonstrated that they could introduce a H_0 bias of order 10%, in the ULDM particle range $10^{-25} \lesssim m/\text{eV} \lesssim 10^{-24}$ and cosmological DM abundance fraction $\alpha_\chi \lesssim 0.15$, compatible with other cosmological constraints [Kob+17; HMG18].

Our study shows that strong galaxy lensing, combined with other cosmological probes like the CMB, could be sensitive, via an H_0 signal, to the presence of a subdominant component of dark matter in the form of ultralight fields or axions. We dub the search of ULDM via this method with the name AxionH0graphy.

To further substantiate these claims, made in Ref. [BT21b], we performed numerical simulations, targeted at understanding the magnitude of the AxionH0graphy effect, i.e. the H_0 bias (or signal) coming from ULDM cores which act as approximate MSD. Our simulations confirm that time-delay cosmography is sensitive to subdominant ULDM components in the very light mass regime. We believe our simulations provide robust results regarding the formation, general scale, and dynamics of ULDM cores. However, the set-up we explore is still simplistic in many important aspects: first, we cannot yet simulate fully self-consistently the dominant component of CDM and stars, that are both important for strong lensing in massive elliptical galaxies. Our simulations either use static external potentials to replace these dominant backgrounds, or a wave implementation of CDM that cannot resolve the dynamics below ~ 1 kpc scales. Second, our simulations start from rather ad-hoc initial conditions, and may miss some features of cosmological structure

formation. More refined studies and simulations may either confirm or refute the theoretical expected effects of subdominant ULDM component on strong gravitational lensing, as presented in Ref. [BT25].

With these caveats in mind, our predictions for the amplitude of the subdominant ULDM effect on strong lensing systems must be understood as crude approximations, motivating more comprehensive analysis. If the AxionH0graphy predictions can be trusted, then ULDM earns another way to be constrained via astrophysics data, in this case via time-delay cosmography.

Chapter 6

ULDM and Stellar Kinematics/Dynamics

In this chapter, we focus on the effects that ULDM has on stellar kinematics and dynamics on galactic systems. We have two main differences which differentiate ULDM with respect to CDM, as far as galactic dynamics is concerned: a cored profile on scales of λ_{dB} in contrast with a cusp [NFW97], and dynamical heating from the soliton and the interference patterns. ULDM-induced dynamical heating of stars tend to increase with time the half-light radius of stellar systems. Its effects are more prominent in dwarf galaxy and ultrafaint dwarf galaxy systems. We investigate such an effect via numerical simulations in Sec. 6.1. On the other hand, cores can influence the rotation curve velocities of stars in galaxies. In Sec. 6.2 we explore this effect by investigating the so-called soliton-halo relation, a relation which supposedly connects characteristics of the halo with the mass of the soliton in the potential well of the galaxy.

6.1 Dynamical heating of stars

The fluctuating density field of ULDM is expected to affect dynamical heating, friction, and relaxation of the stellar body [AL18; COM19; Lan+20; Bar+21; MN19; DK22; SCB20; Yan+24; DC+23]. In the soliton region, quasinormal mode oscillations [GUL04; CSX23] and soliton random walk [LHY21] also contribute to heating. Heating tends to inject energy on stars in galactic systems, making the stellar distribution within a galaxy wider than expected. Estimates of dynamical heating on stars can be obtained from kinetic theory of an homogeneous background of ULDM [Hui+17; BOFT19; BOFT21]. However, realistic halos are not homogeneous, and show dynamical evolution. To characterize the effects of dynamical heating in such halos, we perform numerical simulations that include both the ULDM and the stars, the latter treated as test particles. We here focus on dispersion-dominated dwarfs, in particular the Fornax dwarf galaxy and similar systems.

We explore different initial conditions and ULDM particle masses in the range $10^{-22} \text{ eV} \leq m/\text{eV} \leq 5 \times 10^{-21} \text{ eV}$, and we compare our results with available data from Fornax [WMO09; Wal+09].

The ULDM density profile in our simulations is initialized without a soliton core, but forms it dynamically to a level roughly consistent with the soliton-halo relation discovered in Ref. [Sch+14]. We show that the LOSVD profile approximately follows the steady solution to the collisionless Jeans' equation. However, dynamical heating leads to secular growth of the stellar half-light radius, with a time scale that decreases with increasing m , qualitatively consistent with analytical expectations [BOFT19; BOFT21]. Together, these effects put ULDM in tension with Fornax data for $m \lesssim 10^{-21} \text{ eV}$. In particular, for $m = 10^{-22} \text{ eV}$, the stellar body is contained inside the soliton region and LOSVD can in principle be reconciled with data; however, the dynamical heating timescale is much smaller than the system age of order 10 Gyr. This makes the half-light radius observed today difficult to explain without fine-tuning the initial conditions. For $m = 10^{-21} \text{ eV}$, dynamical heating becomes less important; however, the soliton occupies only the inner part of the stellar body, causing a bump in LOSVD, in tension with the data.

The scaling properties of the SPE, that governs ULDM dynamics [Hui+17], relate our results derived for Fornax to results corresponding to larger m in halos with smaller velocity dispersion and smaller scale size, like Leo II and Carina dwarf galaxies. We perform simulations for these systems and find that the ULDM model tension with data extends up to $m \approx 5 \times 10^{-21}$ eV.

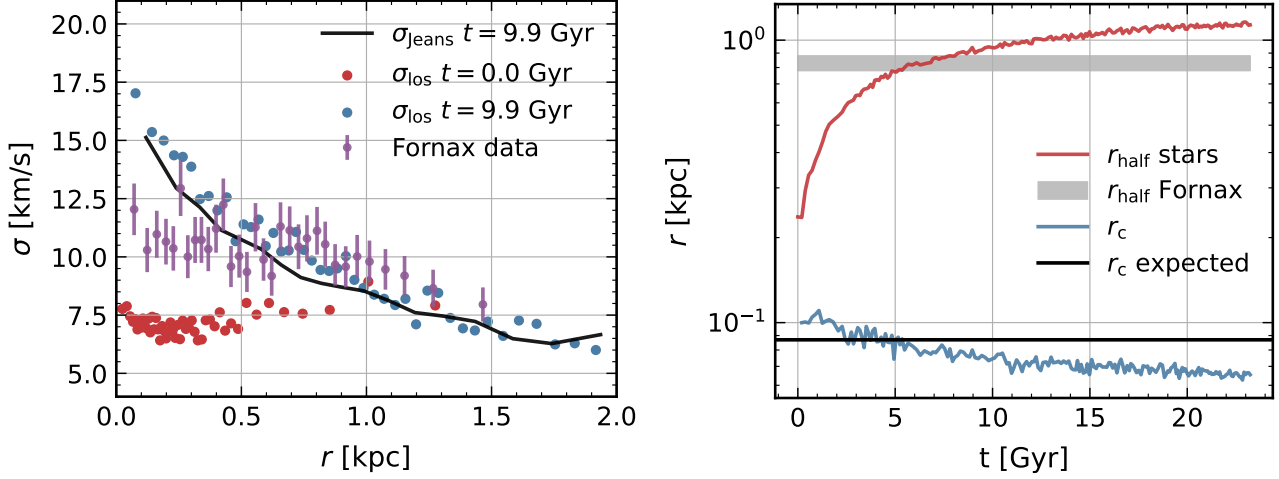


FIGURE 6.1: Simulation results for $m = 1 \times 10^{-21}$ eV, with $L = 12$ kpc. **Left.** LOSVD data for Fornax (purple) is compared to the simulation at time 0 (red) and at time $t \approx 9.9$ Gyr (blue). Every bin has the same number of stars. Black line is the σ_{los} predicted by a Jeans analysis. Notice that LOSVD data are difficult to reconcile with ULDM for $r \lesssim 0.5$ kpc. **Right.** Red lines: half-light radius r_{half} evolution over time. Solid and dotted refer to two simulations, with the same ULDM halo but different initial distribution of stars. Grey band indicates the observed half-light radius of Fornax reported in Tab. I of Ref. [Wal+09], $r_{\text{half}} = 0.668 \pm 0.034$ kpc, while grey dashed line shows the half-light radius from Ref. [Wan+19] (see discussion in Sec. 6.1.6). Blue line shows the soliton core radius r_c . Dashed blue line shows the expectation from the soliton-halo relation of Ref. [Sch+14]. Vertical dashed line highlights $t = 10$ Gyr, which is the typical age of the systems we consider. Notice that fine-tuned initial conditions allow to reproduce the half-light radius from [Wan+19] (albeit not from [Wal+09]).

6.1.1 Simulations

We simulate halos resembling the Fornax, Leo II, and Carina dwarf galaxies. For Fornax we use $M_{200} \approx 10^9 M_\odot$, in a box of length $L = 40$ kpc ($L = 12$ kpc) for $m = 10^{-22}$ eV ($m \gtrsim 10^{-21}$ eV) respectively. For Carina/Leo II, $M_{200} \approx 10^8 M_\odot$, in a box of length $L = 5$ kpc for $m = 5 \times 10^{-21}$ eV. We initialize the ULDM field to a halo with either a cusp NFW [NFW97] or a cored Burkert [Bur15] profile, and the stars to a Plummer profile [Plu11; Dej87], using a version of the Eddington procedure described in Sec. 6.1.2. The ULDM field, together with the stars, are then evolved numerically for about 10 Gyr, the typical age of the stellar population [MR96; Boe+12; Mig97]. The ULDM is evolved using a pseudo-spectral SPE solver, whereas stars are evolved via a second-order leap-frog algorithm, all described in App. A.1.

We run a suite of simulations, varying the initial conditions of the halo and the stellar distribution, in an attempt to make the stellar LOSVD and surface brightness in the simulation as close as possible to observations.

We initialize 10^4 stars in all our simulations. The stars are treated as test particles, and their gravitational potential is neglected. This approximation is a caveat that we discuss further in Sec. 6.1.5.

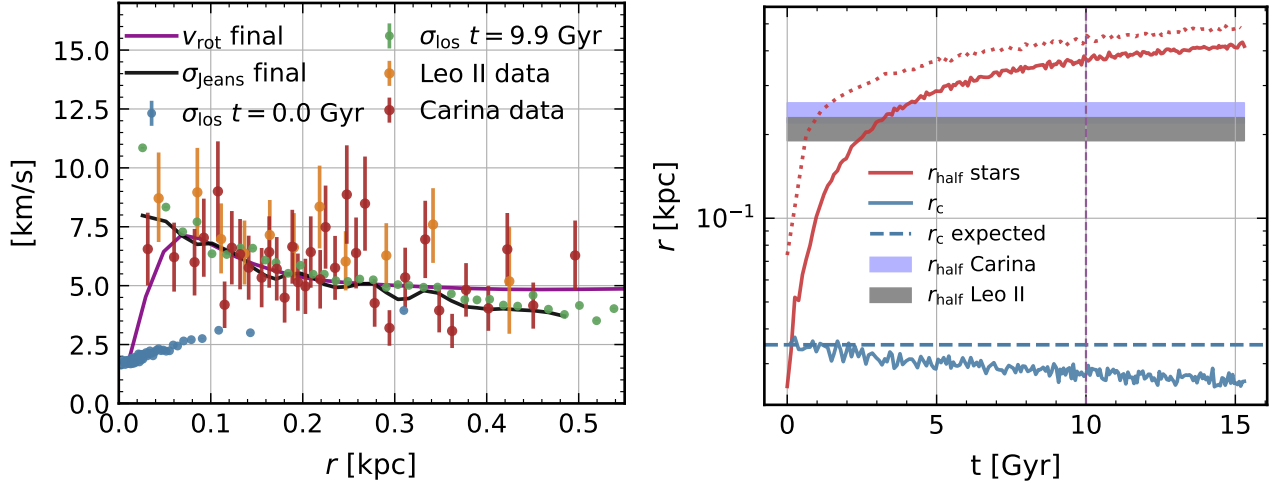


FIGURE 6.2: Simulation of a halo resembling Leo II and Carina, with $m = 5 \times 10^{-21}$ eV, $L = 5$ kpc. Legend as in Fig. 6.1. Note the tension between simulated stellar r_{half} and data. For Leo II we derived LOSVD and r_{half} from Ref. [Koc+07], finding $r_{\text{half}} = 0.21 \pm 0.02$ kpc with uncertainty determined via bootstrap resampling. For Carina we use data from Ref. [Wal+09]; in this case $r_{\text{half}} = 0.24 \pm 0.02$ kpc.

6.1.2 Eddington and Jeans analysis

We initialize halos via the Eddington procedure Eq. (5.25); to implement a non-zero anisotropy, we can use the distribution function [BT08]

$$f(x, \vec{v}) = f_1(\mathcal{E}(x, v)) L^{-2\beta_0}, \quad L = xv \sin \eta, \quad (6.1)$$

where β_0 is a constant anisotropy parameter, η is the angle between the $\vec{x}$ and $\vec{v}$ vector, and $\mathcal{E}$ and Ψ were defined in Eq. (5.27).

If $-0.5 < \beta_0 < 0.5$, one can write [LSL18]

$$f_1(\mathcal{E}) = \frac{2^{\beta_0 - \frac{3}{2}} \Gamma(\frac{3}{2} - \beta_0) \sin((\frac{1}{2} - \beta_0)\pi)}{\pi^{\frac{5}{2}} (\frac{1}{2} - \beta_0) \Gamma(1 - \beta_0)} \frac{d}{d\mathcal{E}} \int_0^{\sqrt{\mathcal{E}}} d\Psi \frac{d(r^{2\beta_0} \rho)}{d\Psi} (\mathcal{E} - \Psi)^{-\frac{1}{2} + \beta_0}. \quad (6.2)$$

In the case of $\beta_0 = 0$, Eq. (6.2) reduces to Eq. (5.26).

Stars are initialized using the same Eddington formalism, where ρ is the density profile of stars (e.g. Plummer) and Φ is sourced by the dark matter halo. Once f^{stars} is computed, we can generate phase space coordinates for stars as follows. Choose a maximum radius R_{max} (typically, half the half length of the simulation box); then, the probability that a stars sits at a distance x is

$$p(x) = \frac{4\pi\rho(x)x^2}{M(R_{\text{max}})}, \quad (6.3)$$

and x coordinates are sampled using the cumulative of $p(x)$. The probability that a star has a velocity v , given its position x , reads

$$p(v|x) = \frac{p(v, x)}{p(x)} = \frac{f_1(\mathcal{E}(x, v)) v^{2-2\beta_0}}{\int dv f_1(\mathcal{E}(x, v)) v^{2-2\beta_0}}, \quad (6.4)$$

and again v coordinates are sampled via its cumulative. For the angle η , we use

$$p(\eta) = \frac{\sin^{1-2\beta_0} \eta}{\int d\eta \sin^{1-2\beta_0} \eta} . \quad (6.5)$$

The θ, ϕ angle for the $\vec{x}$ spherical coordinates are sampled uniformly on the unit sphere; the components of the vector v in spherical coordinates then read

$$v_r = v \cos \eta , \quad v_\theta = v_r \cos \psi , \quad v_\phi = v_r \sin \psi , \quad (6.6)$$

where ψ is sampled uniformly on the unit circle. Then the Cartesian coordinates of $\vec{v}$ are found via

$$\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ \cos \theta & -\sin \theta & 0 \end{pmatrix} \begin{pmatrix} v_r \\ v_\theta \\ v_\phi \end{pmatrix} . \quad (6.7)$$

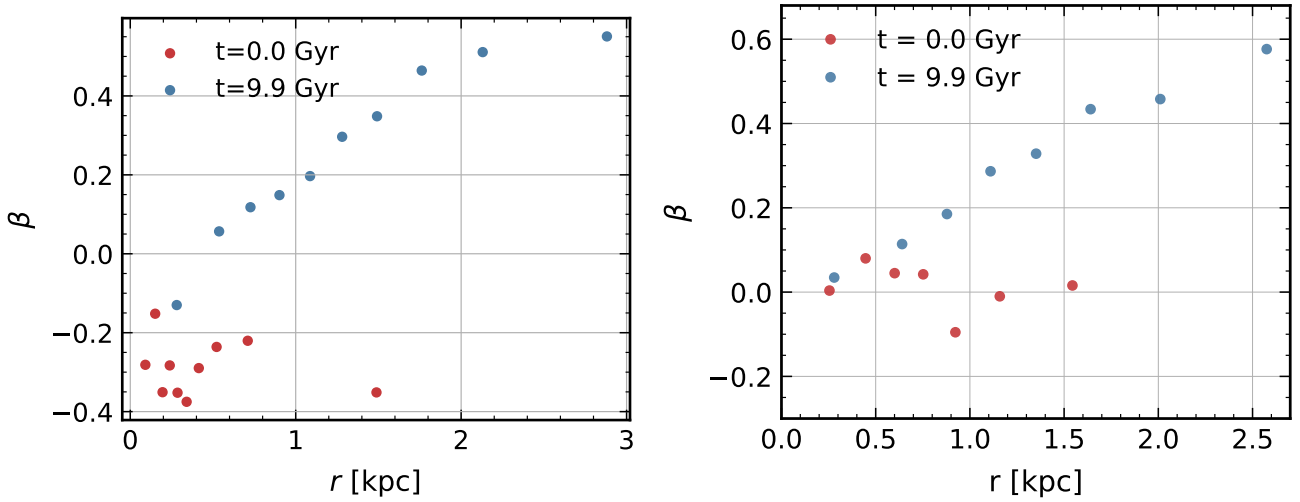


FIGURE 6.3: Evolution of the anisotropy parameter for a simulation of a halo initialized as NFW, with $L = 12$ kpc, $m = 10^{-21}$ eV. **Left:** initial $\beta = -0.3$. **Right:** initial $\beta = 0$. β evolves towards positive values due to ULDM dynamical heating.

We have the simulation data $\vec{r}_i(t), \vec{v}_i(t)$ for the stars (labeled with the subscript i), where $\vec{r}$ is computed relative to the center of mass of the stars. Given a time t at which we want to compute velocity dispersions, we consider only the stars for which $\max_{t' < t} (|\vec{r}_i(t')|) < R_{\max}$, to avoid boundary artifacts. We then choose a random line of sight $\hat{l}$, we look for stars at a certain distance r_{proj} , where

$$r_{i,\text{proj}} = |\vec{r}_i - (\vec{r}_i \cdot \hat{l})\hat{l}| , \quad (6.8)$$

is the distance projected on the surface perpendicular to the random line of sight, and compute the variance of the projected velocity $\vec{v}_i \cdot \hat{l}$ of all the stars falling into the shell of thickness dr_{proj} . This translates into

$$\sigma_{\text{los}}(r) = \sqrt{\text{Var}[\vec{v}(r) \cdot \hat{l}]} . \quad (6.9)$$

When comparing $\sigma_{\text{los}}(r)$ with the result predicted by a spherically symmetric Jeans analysis, we numerically compute $\rho_*(r)$, the radially averaged star density, $\Sigma_*(r_{\text{proj}})$, the star column density along the line of sight $\hat{l}$ radially averaged over the surface perpendicular to $\hat{l}$, and $M(r)$, the dark matter mass inside the radius r . We then use the line of sight velocity dispersion Eq. (1.92).

6.1.3 Results

We compare the results of our simulations with observational data of Fornax [Wal+09], Leo II and Carina galaxies [Koc+07], which we describe in Sec. 6.1.6. Fig. 6.1 shows an example of simulation results and comparison with data, for Fornax. The left panel shows the stellar LOSVD in the simulation at time zero (red points) and after $t \approx 9.9$ Gyr (blue points), compared with data (purple). The black line shows the prediction of a Jeans analysis using as input the true ULDM radially-averaged halo mass profile $M(r)$, stellar density $\rho_*(r)$, column density $\Sigma_*(r_{\text{proj}})$, and anisotropy parameter $\beta(r)$ from the simulation. The right panel shows the stellar half-light radius, which exhibits secular growth. We also show the central soliton core size, which evolves to become more massive and more compact, consistent with the relation $M_{\text{sol}} \propto 1/r_{\text{sol}}$ [Hui+17].

We conducted a suite of 38 simulations, varying the DM mass, density profile type (NFW or Burkert) and scale radius, the initial stellar half-light radius, and the initial stellar velocity anisotropy parameter β . We now summarize the results of this survey.

For $m = 10^{-21}$ eV (example in Fig. 6.1), the LOSVD peak in the soliton region, along with the flat tail outside the soliton region, are difficult to reconcile with the data. A solution that fits the tail, overshoots the data in the inner region, and vice-versa. One might think that a negative anisotropy β parameter could help in lowering the inner LOSVD peak; however, ULDM heating injects radial anisotropy in the star distribution. Our simulations show indeed that β tends to become positive on a dynamical heating timescale. Fig. 6.3 shows a sample halo initialized as an NFW with $L = 12$ kpc and $m = 10^{-21}$ eV. In Fig. 6.3 left we initialize $\beta = -0.3$, while for comparison, in Fig. 6.3 right, we initialize $\beta = 0$. As can be seen, both simulations evolve and reach similar values of $\beta \approx +0.5$ by $t \approx 10$ Gyr.

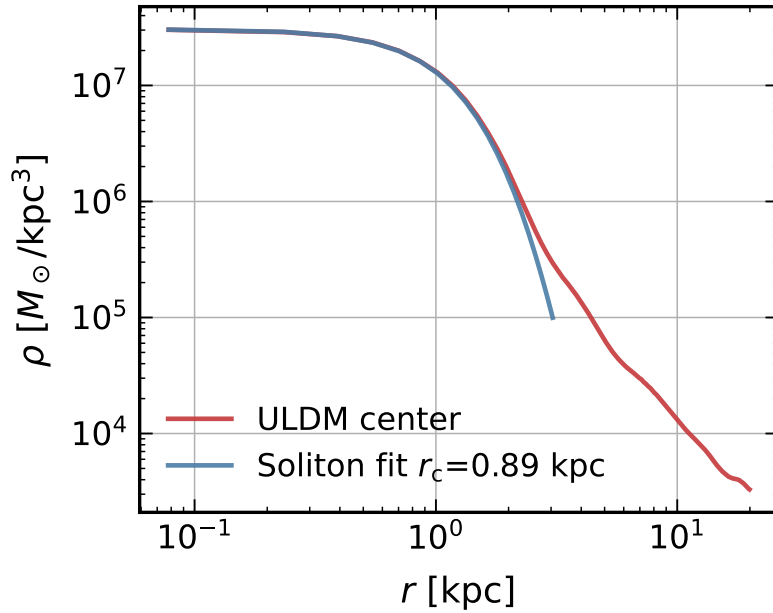


FIGURE 6.4: ULDM density profile (red curve) for a simulation with $L = 40$ kpc, $m = 1 \times 10^{-22}$ eV, evolved for 10 Gyr. We also show (blue curve) the soliton fit using Eq. (3.25); the resulting core radius is $r_c = 0.89$ kpc, close to the value found in Ref. [Sch+14], $r_{\text{expected}} = 0.98$ kpc.

This suggests that positive β is an attractor for stars in an ULDM halo (see also [DC+23]), in agreement with the intuition that stars are “kicked out” from the center of the system into eccentric orbits.

Although the stellar distribution is not stationary, due to the ULDM dynamical heating, a stationary Jeans analysis continues to describe the system fairly well.

This result can be understood by comparing the heating timescale, derived in the idealized case of homogeneous isotropic ULDM background [BOFT19; BOFT21]

$$T_{\text{heat}} = \frac{3\sigma^6 m^3}{16\pi^2 G^2 \rho^2 \ln \Lambda_C} \approx 0.48 \text{ Gyr} \left(\frac{m}{10^{-21} \text{ eV}} \right)^3 \left(\frac{\sigma}{10 \text{ km s}^{-1}} \right)^6 \left(\frac{0.01 \text{ M}_\odot / \text{pc}^3}{\rho} \right)^2, \quad (6.10)$$

with the dynamical time of the system,

$$T_{\text{dyn}} = \frac{R}{v} \approx 0.1 \text{ Gyr} \left(\frac{R}{1 \text{ kpc}} \right) \left(\frac{10 \text{ km s}^{-1}}{v} \right), \quad (6.11)$$

which shows $T_{\text{dyn}} \lesssim T_{\text{heat}}$ for $m = 10^{-21} \text{ eV}$ in a Fornax-like halo. Note, however, that our simulations are not homogeneous or isotropic, and naturally contain additional effects like soliton random walk [Cho+21; SCB20], so we expect Eq. (6.10) to be only qualitatively valid. In fact, we stress that we do not use Eq. 6.10 in any numerical computation, merely, here, to obtain a rough analytical understanding of the relevant time scale.

For $m = 10^{-22} \text{ eV}$ (example in Fig. 6.6), the whole stellar body is inside the soliton region. It is then possible to find a soliton solution which fits the LOSVD and surface brightness data: this was a main result of Ref. [SCB14]. However, this solution is a brief unstable transient. Because the heating timescale for this value of m is fast, the LOSVD grows with time and exceeds the observed range.

This increase can be traced to the rapid stellar half-light radius r_{half} growth, which also makes it difficult to justify the half-light radius seen in the data today. (See, however, the discussion of our neglect of stellar self-gravity, expanded in Sec. 6.1.5: this is not a major effect for stellar radius observed today, but can become important if the stellar profile was much more compact in the past.)

For $m > 10^{-21} \text{ eV}$, the timescale for the formation of the soliton in a galaxy like Fornax exceeds 10 Gyr, reducing the soliton mass today and its impact on stellar LOSVD. Fig. 6.5 demonstrates this point with $m = 5 \times 10^{-21} \text{ eV}$. We also find that for $m \gtrsim 5 \times 10^{-21} \text{ eV}$ dynamical heating is negligible. An adequate fit of the observational data of Fornax can be obtained in this case; this would be, in effect, simply a CDM-like model.

Considering also Leo II and Carina, we show a sample result of our simulations in Fig. 6.2, for $m = 5 \times 10^{-21} \text{ eV}$. Contrary (and complementary) to Fig. 6.1 we see that in this case the LOSVD data can be reproduced rather well. However, the smaller velocities in these systems make dynamical heating faster, causing rapid growth of r_{half} that is difficult to reconcile with the observed value. We conclude that $m \leq 5 \times 10^{-21} \text{ eV}$ is in tension with data.

6.1.4 Comparison with previous works.

We find it useful to compare our analysis with related works. Our bound $m \geq 5 \times 10^{-21} \text{ eV}$ is somewhat stronger than that found in [Bar+18; Bar+19; BBS22] from disk galaxies. We note that while we did not explore $m < 10^{-22} \text{ eV}$, complimentary bounds from cosmology disfavor that regime [Ir17; Arm+17; Kob+17; Leo+19; Lag+22].

Ref. [DC+23] conducted similar (somewhat higher resolution) simulations of test-particle stars in an ULDM dwarf galaxy halo. Their results confirm ULDM-induced dynamical heating, leading to an increase of both the half-light radius and the β parameter of the stellar system. However, Ref. [DC+23] considered only a single mass, $m = 8 \times 10^{-23} \text{ eV}$, and did not report a comparison with observational data. In contrast, we explored a range of m and compared our results to data.

Ref. [Zim+24] used Leo II data to perform a Jeans analysis, varying the ULDM density profile and carefully comparing the results with data from [Spe+17]. They deduced the bound $m \gtrsim 2.2 \times 10^{-21} \text{ eV}$,

similar to the bound we inferred. However, the work was not based on ULDM simulations but rather on a parametric eigenstate expansion of the ULDM wave function. Consequently, Ref. [Zim+24] did not provide information about secular evolution, which our study suggests to be a significant effect of ULDM in this mass range.

Finally, Ref.[DK22] (see also Ref.[MN19], and the related discussion in [CSC21]) argued for the exclusion of $m \lesssim 3 \times 10^{-19}$ eV, based on dynamical heating estimates and noting the stellar scale sizes and LOSVD in Segue 1 and Segue 2, but without self-consistent numerical simulations of the ULDM+stellar system. Our work here, based on simulations and applied to a different set of galaxies, is consistent with the conclusions of Ref. [DK22] in the range of m we simulated. (As an aside, the caveat in our analysis, due to omitting stellar self-gravity, applies also to Ref.[DK22], [MN19], and [CSC21].)

6.1.5 When is stellar self-gravity important?

Our analysis, treating ULDM and stars dynamically in tandem and comparing the results directly to observational data, is a step forward in comparison to previous work. However, a remaining limitation is the treatment of stars as test particles. This approximation seems reasonable given the dark-to-luminous mass ratios observed in dwarf spheroidal galaxies today, but it can fail if, in the past, the stellar population was more concentrated. For Fornax, Ref. [BSN15] quotes a dark-to-luminous mass ratio of about 5–6 (16–18) within 1.6 (3) kpc [BSN15]. To illustrate how the dark-to-luminous mass ratio scales with the parameters of interest and within the radius of the galaxy, let us assume an NFW profile for DM and a Plummer profile [Plu11; Dej87] for the stellar distribution.

The enclosed NFW mass profile is given by

$$M_{\text{NFW}}(< r) = 4\pi R_s^3 \rho_s \left[\ln\left(1 + \frac{r}{R_s}\right) - \frac{r}{r + R_s} \right], \quad (6.12)$$

while the enclosed Plummer profile is

$$M_{\text{Plummer}}(< r) = M_0^* \left(\frac{r}{r_{\text{half}}} \right)^3 \frac{1}{\left(1 + \frac{r}{r_{\text{half}}}\right)^{3/2}}, \quad (6.13)$$

where M_0^* is a normalization constant that accounts for the total number of stars, and r_{half} is their half-light radius.

In Fig. 6.7, we show the NFW mass profile (blue solid line) for $R_s = 1.5$ kpc and $\rho_s = 0.01 M_\odot/\text{pc}^3$, which fits well the LOSVD data at large radii, alongside three different stellar mass profiles (black lines). The solid black curve corresponds to a Plummer profile with $r_{\text{half}} = 0.7$ kpc, chosen to match the latest determination of Ref. [Wal+09] adopted in the main text, and a normalization M_0^* set so that

$$\frac{M_{\text{NFW}}(< 1.6 \text{ kpc})}{M_{\text{Plummer}}(< 1.6 \text{ kpc})} = 5. \quad (6.14)$$

In this case DM dominates by a factor of a few inside all the radii of interest. Then, the dashed and dotted curves keep the same total stellar mass normalization (integrated to infinity) but reduce r_{half} to 0.4 kpc and 0.2 kpc, respectively.

We conclude that if the same number of stars were compressed by a factor $\sim 3 - 4$ in their radial distribution, then stellar self-gravity would become significant, and eventually dominate the dynamics, rendering our simulations inadequate in that regime. This means that our simulations of the early evolution in some of the stellar half-light radius curves in Figs. 6.1, 6.6, and 6.2 are not fully consistent. The simulations should evolve into control once the stellar populations expand, within a few Gyr, and well before $t \approx 10$ Gyr in all cases.

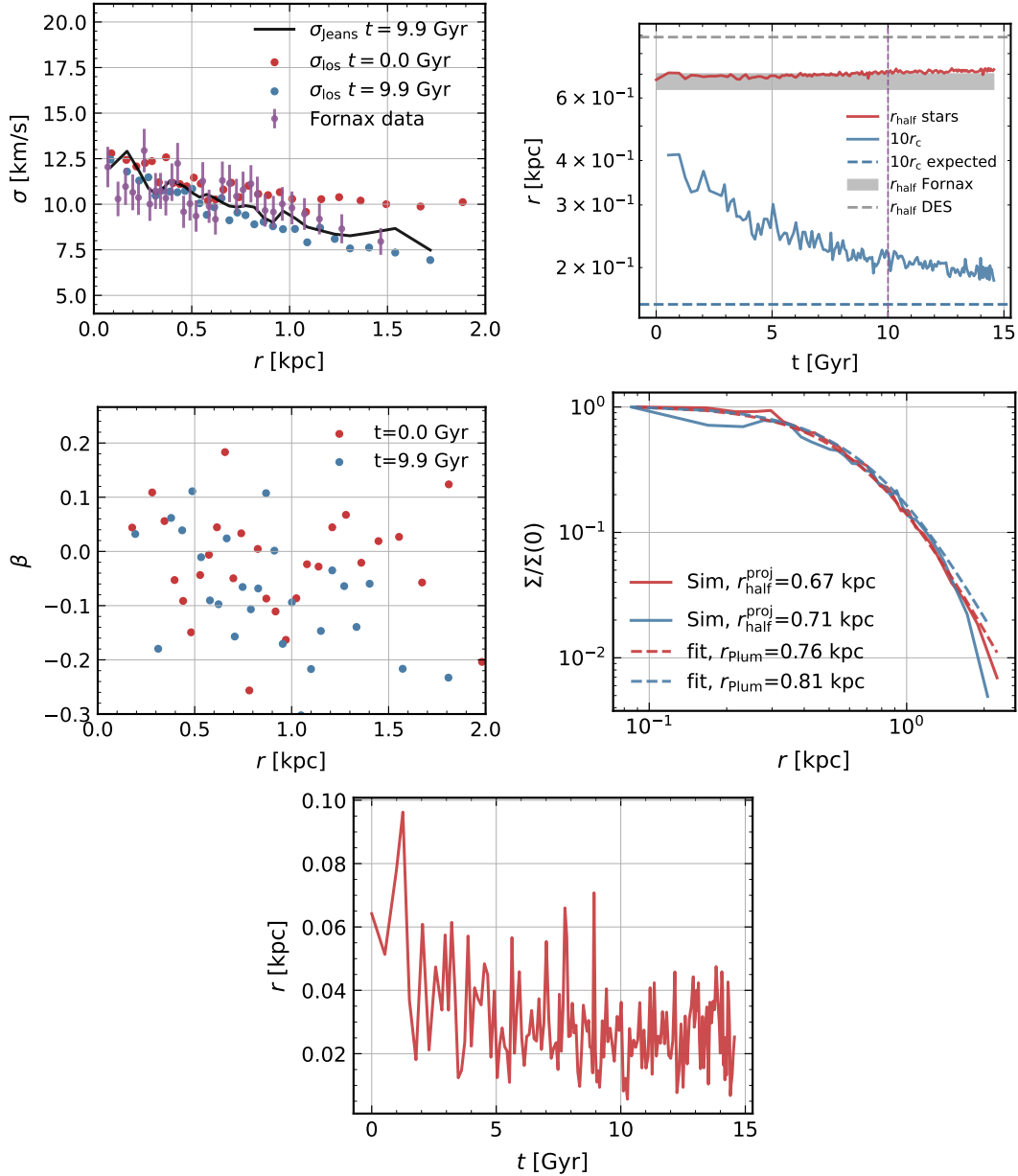


FIGURE 6.5: Simulation initialized as an NFW halo, with $L = 8$ kpc, $m = 5 \times 10^{-21}$ eV. This is an example where dynamical heating is not significant, as we approach the “CDM limit” of ULDM. **Upper left.** LOSVD data for Fornax (purple) is compared to the simulation result at time 0 (red) and at time $t \approx 9.9$ Gyr (blue). Black line is the σ_{los} predicted by a Jeans analysis. **Upper right.** Red line: stellar r_{half} evolution in time. Grey band shows the observed r_{half} from Ref. [Wal+09], grey dashed line shows r_{half} from Ref. [Wan+19]. Blue line shows the soliton core radius r_c . Dashed blue line shows the value of r_c expected from the soliton-halo relation of Ref. [Sch+14]. Vertical dashed line denotes $t = 10$ Gyr. **Center left.** Anisotropy parameter β for $t = 0$ (red) and $t = 9.9$ Gyr (blue). **Center right.** Radially averaged stellar column density in the simulation for $t = 0$ (red) and $t = 9.9$ Gyr (blue), compared with a Plummer profile fit (dashed). We quote the Plummer scale parameter of the fit r_{plum} , together with $r_{\text{half}}^{\text{proj}}$ from the simulation. **Bottom.** Displacement between the stars center of mass and the maximum ULDM density point, which approximately coincides with the soliton.

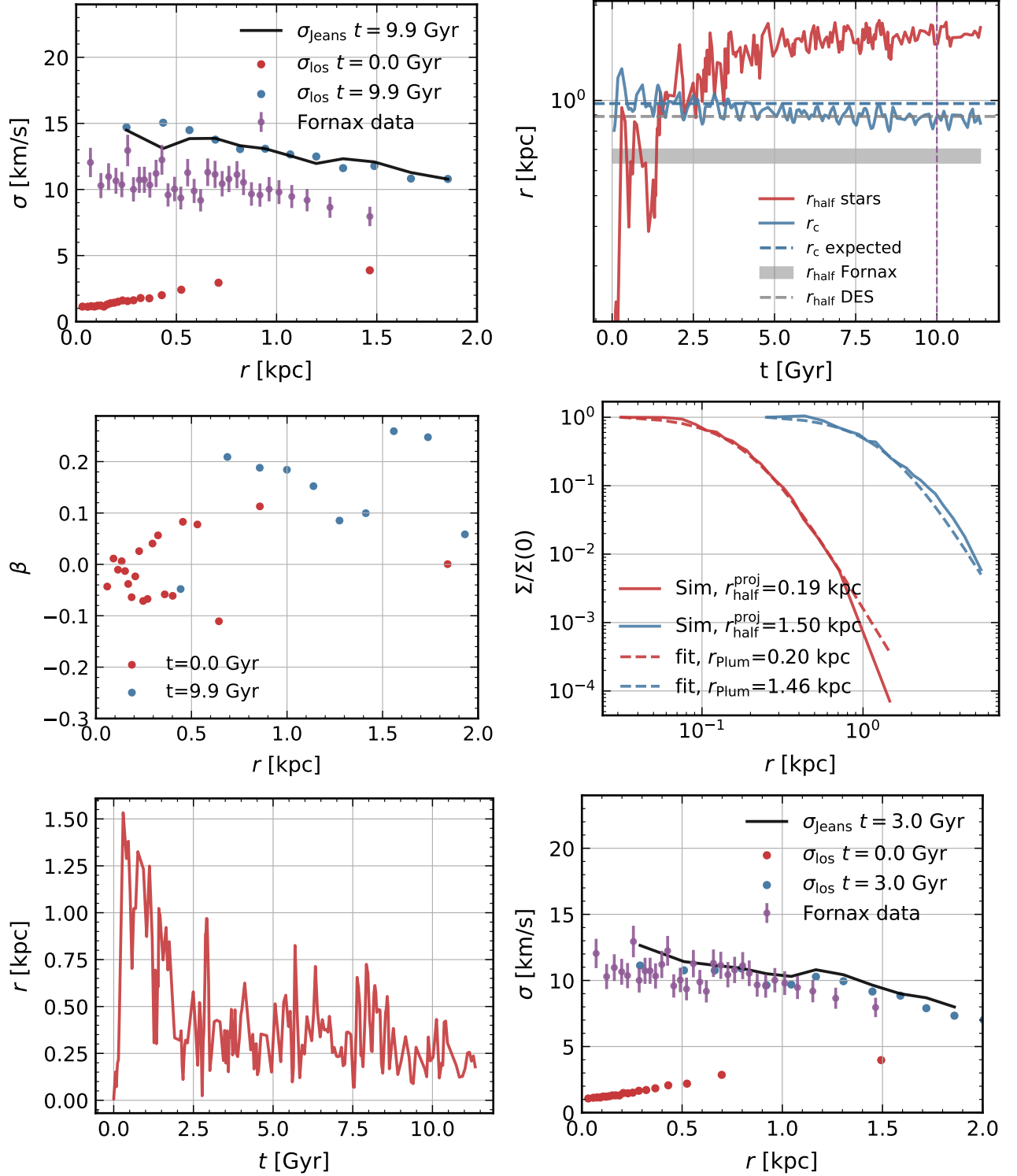


FIGURE 6.6: Simulation for a Fornax-like system for $m = 10^{-22}$ eV, with $L = 40$ kpc. Panels explanation is the same as in Fig. 6.5, with the addition of the bottom-right panel, which is the same as the upper-left panel, but for the $t = 3$ Gyr snapshot.

6.1.6 Dwarf galaxy data

For LOSVD data we have analyzed the following repository from Ref. [Wal+09]. This catalog has spectroscopic data for individual stars observed as part of the Michigan/MIKE Fiber System survey of four

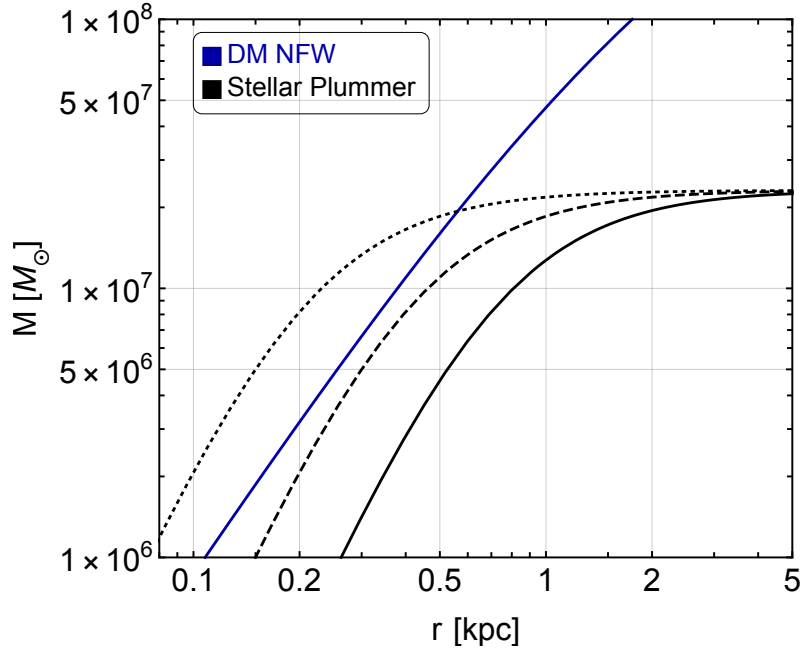


FIGURE 6.7: NFW mass profile for DM with $R_s = 1.5$ kpc and $\rho_s = 0.01 M_\odot/\text{pc}^3$ (blue curve) and different stellar Plummer mass profiles (black curves). For the stellar profiles we have a Plummer profile with $r_{\text{half}} = 0.7, 0.4, 0.2$ kpc, for the solid, dashed, and dotted curves, respectively.

dwarf spheroidal galaxies: Carina, Fornax, Sculptor, and Sextans. For Fornax, following [Wal+09] we selected stars with membership probability of at least 95%. From these, we extracted stellar velocities, binned the data by radial distance from the galaxy center (ensuring each bin contained the same number of stars), and calculated the velocity dispersion for each bin.

We note that the choice of membership probability threshold can impact the results. Fig. 6.8 illustrates the LOSVD profiles for Fornax with two different membership probability thresholds: 95% (red data points) and 99% (blue data points). In both cases, each bin contains 60 stars. The results show that the stricter 99% threshold substantially affects the inferred LOSVD profile. This cut also reduces the number of stars in the sample from 1920 to 1140.

In previous sections we quote the half-light radius reported in Tab. I of Ref. [Wal+09], $r_{\text{half}} = 0.668 \pm 0.034$ kpc, which is also based on the data from the Michigan/MIKE Fiber System survey. (Ref. [Wal+09] has an Erratum which affects the half-light radius. See also the latest arXiv version of the paper.) The DES Collaboration [Wan+19] found a higher value from their survey $r_{\text{half}} = 0.89$ kpc¹, and we show that too in relevant plots for reference. It is worth noting that Table II of Ref. [Wan+19] shows that r_{half} can vary by a factor of two between, for example, the horizontal branch and the blue plume star subsamples. The data for Carina was also obtained from Ref. [Wal+09] with membership probability cut of 95%. The half-light radius is $r_{\text{half}} = 0.24 \pm 0.02$ kpc, where the uncertainty was determined via bootstrap resampling [Efr79].

Finally, we have analyzed data for the Leo II dwarf spheroidal galaxy, available in this repository. This dataset provides the LOSVD derived from 200 stars measured using the FLAMES/GIRAFFE spectrograph at the European Southern Observatory in Chile [Koc+07]. We applied the same cut as in Ref. [Koc+07], retaining only stars within 3σ of the mean radial velocity. This cut reduced the number of stars to 171. These were then binned into groups of 12 stars each, reproducing fairly well Fig.7

¹Another study, based on older data [McC12], reports $r_{\text{half}} = 0.710 \pm 0.077$ kpc (see their Table III).

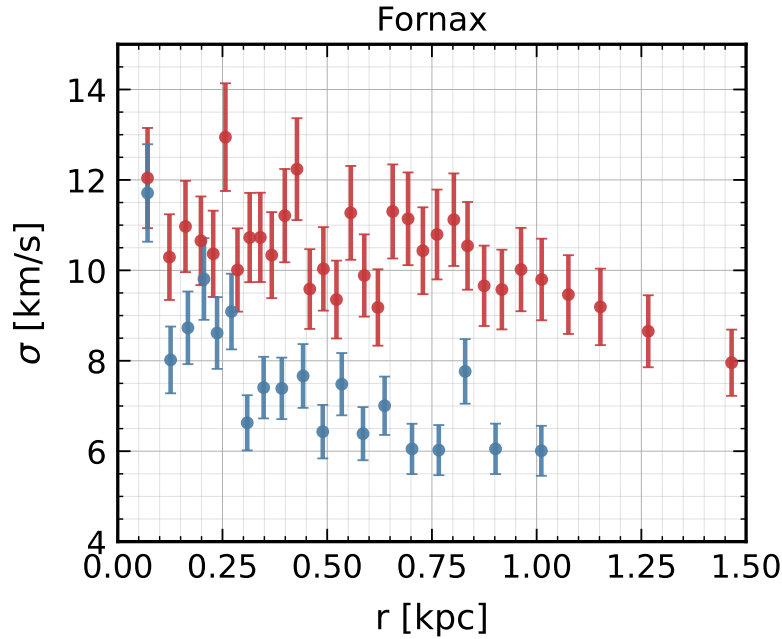


FIGURE 6.8: LOSVD profile for Fornax. The red (blue) data points correspond to a 95% (99%) membership probability cut.

(left panel) from Ref. [Koc+07]. Using the same data we also derived the observed half light radius for this population to be $r_{\text{half}} = 0.21 \pm 0.02$ kpc.

6.1.7 Summary of our results of ULDM dynamical heating in dwarf galaxies

In this section, we focused on how stellar kinematics and surface brightness data of dwarf spheroidal galaxies, in particular Fornax, Carina and Leo II, compare with a ULDM universe.

For $m = 10^{-21}$ eV, stellar velocity dispersion data from Fornax reveal a tension caused by the soliton core. At $m = 10^{-22}$ eV, a stationary Jeans analysis can fit the data, but dynamical heating renders this fit transient. The resulting stellar LOSVD and half-light radius evolve significantly, requiring fine-tuned initial conditions to match observations.

Galaxies with lower velocity dispersion and smaller scale radius are sensitive to higher values of m . We showed that for $m = 5 \times 10^{-21}$ eV, the observed half-light radius of the Carina and Leo II dwarf galaxies is difficult to reconcile with the rapid ULDM heating-driven evolution, that tends to predict larger r_{half} . Our analysis suggests that Leo II and Carina (and other low-dispersion dwarfs) may allow to probe ULDM to larger values of m . We postpone this study to future work.

We conclude with a word of caution regarding a caveat in our analysis. In our simulations, stars are treated as test particles. This leaves open the possibility, especially relevant if the stellar body was significantly more compact in the past, that stellar self-gravity and stellar feedback on the ULDM halo could mitigate the increase in r_{half} . In future work, we plan to incorporate N -body interactions and stellar feedback on the ULDM halo into our simulations.

6.2 Rotation curves and the soliton-halo relation

The formation of cores in the potential well of galaxies in the ULDM scenario has phenomenological consequences on the rotation curve of stars. In particular, a typical ULDM halo rotation curve has two

peaks: one inner peak induced by the soliton, and one outer peak induced by the NFW halo. To understand the magnitude of such an effect, a natural question is whether a soliton actually forms within the lifetime of a galaxy and, if so, what is its expected mass for a given host galaxy halo. Dynamical relaxation estimates of the timescale for soliton formation are consistent with the results of simulations that use “noise” initial conditions, which are designed to be in the kinetic regime [BOFT19; BOFT21; LPT18; Dmi+24]. Meanwhile, simulations with cosmological initial conditions – notably those of Ref. [SCB14; MS21] – suggest that solitons in cosmological halos may form more rapidly than predicted by kinetic theory estimates (the impact of baryons has been studied in Ref. [VSN20]). Regarding the expected soliton mass, the cosmological simulations in Ref. [SCB14; Sch+14] (see also Ref. [Lia+24]) provided numerical evidence for a simple relation between the soliton mass and the host halo mass and energy. Such a relation translates into the simple statement that the inner rotation curve peak induced by the soliton has the same velocity value as the outer rotation curve peak induced by the NFW halo. Such rotation curve behaviour can be tested against observations of rotation curves, like the work in Refs. [Bar+18; BBS22], making its determination of phenomenological interest. Ref. [Sch+14] also suggested that the relation may represent an attractor of the equations of motion, supporting this point via simulations with different initial conditions. Many other investigations of the soliton-halo relation have subsequently appeared in the literature, reporting varying levels of agreement [Sch+14; SNE16; Du+17; Moc+17; Cha+22; Zag+23; Lag+24; Lia+24; MTT24; YLH22]. Here, we present a new perspective on the problem. A summary of our main understanding is as follows.

The soliton-halo relation found in Ref. [Sch+14] was fit in that paper by the expression²

$$M_{\text{sol}} = 4.2 \left(\frac{|E_{\text{tot}}|}{M} \right)^{1/2} \frac{1}{Gm}, \quad (6.15)$$

where M_{sol} is the soliton mass. In a cosmological setting E_{tot} and M are the total energy and the virial mass of a halo, including the soliton. In flat-space simulations E_{tot} and M are the total energy and the total mass in the simulation box (both of these are conserved quantities), which are a proxy for the halo properties in a cosmological system, assuming most of the simulated volume is virialized.

The simulations used to obtain Eq. (6.15) employed toy initial conditions (multiple solitons starting at rest); however, Ref. [Sch+14] noted that the result was consistent with cosmological simulations [SCB14; Cha+22]. The prefactor ≈ 4.2 in Eq. (6.15) was obtained in [Sch+14] by fitting the numerical data.³ The exponent $1/2$ on $|E_{\text{tot}}|/M$ was also confirmed numerically, with insight from cosmological simulations. We will call Eq. (6.15) the $1/2$ relation.

Ref. [Bar+18] pointed out that the $1/2$ relation is directly equivalent to the relation

$$\frac{E_{\text{sol}}}{M_{\text{sol}}} = \frac{E_{\text{tot}}}{M} \quad (\text{equivalent to Eq. (1)}) . \quad (6.16)$$

This follows because the soliton by itself satisfies the relation $M_{\text{sol}} \approx 4.2 \left(\frac{|E_{\text{sol}}|}{M_{\text{sol}}} \right)^{1/2} \frac{1}{Gm}$, as can be verified by direct calculation of the soliton solution. We note that it is unlikely that a soliton would have a mass significantly smaller than that determined by the $1/2$ relation. This is suggested by the results of Ref. [CSX24], which investigated the evaporation and growth of a soliton in a homogeneous “noisy” background, showing that in such a system solitons with masses smaller by more than a factor of approximately 3 than those predicted by Eq. (6.16) would evaporate due to scattering with the background particles. We return to this point in more detail later on.

²See Eq. (5) and Fig. 4 *there*; we restore natural units to their expression. Ref. [Sch+14] referred to M_c rather than M_{sol} , where M_c is the soliton mass enclosed in the the core radius r_c , at which the density drops to half its maximum value. The relation is $M_{\text{sol}} \approx 4.2M_c$.

³Since Ref. [Sch+14] phrases Eq. (6.15) in terms of M_c , the prefactor in Eq. (5) *there* is $\alpha = 4.2M_c/M_{\text{sol}} \approx 1$.

Another soliton-halo relation was proposed in Ref. [Moc+17]. We call this the 1/3 relation since it can be rewritten as

$$M_{\text{sol}} = 2.6 \left(\frac{|E_{\text{tot}}|}{G^2 m^2} \right)^{1/3}. \quad (6.17)$$

In fact, if the simulated system is approximately virialized, then Eq.(6.17) can only be an upper bound on the soliton mass. This can easily be understood by the fact that, as pointed out in Ref. [Bar+18], Eq. (6.17) is directly equivalent to

$$E_{\text{sol}} = E_{\text{tot}}, \quad (6.18)$$

because an expression similar to Eq. (6.17) holds for a single soliton: $M_{\text{sol}} \approx 2.64 \left(\frac{|E_{\text{sol}}|}{G^2 m^2} \right)^{1/3}$ (see also App. A.2). A soliton more massive than that predicted by the 1/3 relation would have more energy than the entire halo (in absolute value), which is an impossibility for bound virialized systems. The 1/3 upper bound may be violated if a flat-space simulation contains unbound particles since these would contribute positively to E_{tot} ; the bound however holds if E_{tot} is restricted to the bound particles.

To test the picture above, we performed simulations of ULDM in flat space-time and studied the soliton-halo configurations arising from the evolution of different types of initial conditions. The results, collected in Fig. 6.9, are consistent with our expectations: the solitons form not far from (and usually below) Eq. (6.16), and thus from the 1/2 relation. Subsequently, the solitons accrete more mass, but do not surpass the 1/3 relation. Fig. 6.11, which uses variables that allow both relations to be represented on a single plot, demonstrates this point.

In Sec. 6.3, we compare with other works and show that their data is not clearly inconsistent with the band enclosing the 1/2 and 1/3 relations (after accounting for an arbitrary additive constant present in the definition of the potential energy for finite box simulations).

If our results can be translated to a cosmological setting (which may very well be the case, see Fig. 6.19), ULDM bounds based on the 1/2 relation may be considered conservative when applied to astrophysical systems, provided that the soliton formation time is not too large compared to the age of the system. Assuming the kinetic theory relaxation calculation gives the relevant timescale, the formation of the soliton limits the applicability of the soliton-halo relation in galactic systems like dwarf galaxies to ULDM masses $m \lesssim 10^{-20}$ eV [Dmi+24].

6.2.1 Simulation setup

We consider systems in boxes with periodic boundary conditions on grids with 256^3 or 512^3 points, and evolve them numerically with a standard pseudo-spectral algorithm (see e.g. [LPT18]). Key quantities are the energy of a soliton $E_{\text{sol}} = E_{\text{kin,sol}} + E_{\text{pot,sol}}$ and of the halo $E_{\text{tot}} = E_{\text{kin}} + E_{\text{pot}}$, where $E_{\text{kin(sol)}}$ and $E_{\text{pot(sol)}}$ are the kinetic and potential energies. As in all finite-box (non-GR) simulations, the potential energy E_{pot} is defined only up to an additive constant, and the naive evaluation of this is unphysical as it depends on the box size. We choose this constant such that the gravitational potential of the resulting halo matches the behavior that would occur in the infinite-volume limit ($\propto 1/r$), see App. A.1. Where possible, we present our results using E_{kin} (which is independent of this constant and the uncertainty associated with its determination), as we discuss further in Sec. 6.2.4.

Upon redefining space, time and density as $x \rightarrow x/m$, $t \rightarrow t/m$, $\rho \rightarrow m^2/(4\pi G)\rho$ respectively, the SPE do not depend on m . When showing results we however set $m = 10^{-21}$ eV. Additionally, the SPE retain a residual scaling symmetry (rescaling the field, distances, times and potentials, see App. A.1). We use this to fix the box length to $L = 10$ kpc; this choice is useful to connect the DM halos in our simulations to observed galactic halos, which have scale radii of $O(10 \text{ kpc})$. In combination with our choice of m , this also fixes the units of time and density, expressed in Gyr and $M_{\odot}/\text{kpc}^3$ respectively.

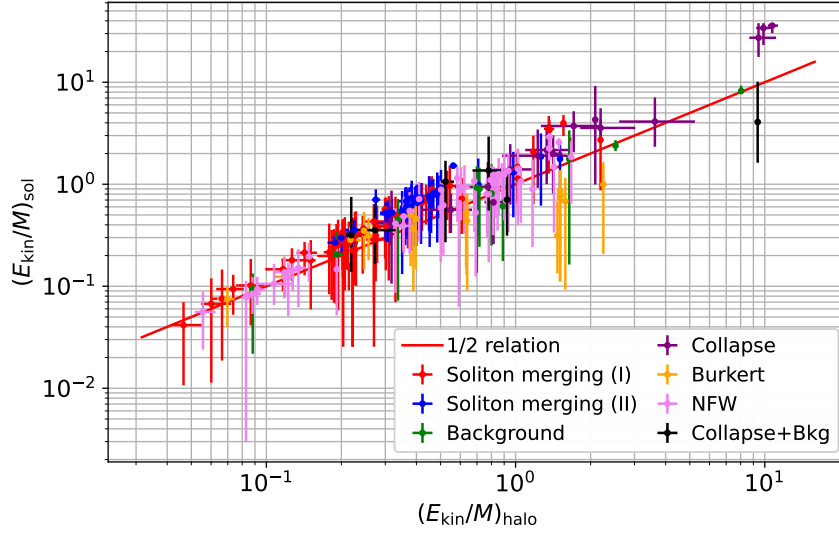


FIGURE 6.9: The kinetic energy-to-mass ratio of solitons compared to that of their halos, $(E_{\text{kin}}/M)_{\text{sol}}$ and $(E_{\text{kin}}/M)_{\text{halo}}$ respectively. Data points are the results from simulations with all the types of initial conditions that we tested, see Section 6.2.2. The red line is the 1/2 relation in Eq. (6.16), i.e. $(E_{\text{kin}}/M)_{\text{sol}} = (E_{\text{kin}}/M)_{\text{halo}}$. After the soliton halo-system forms, both $(E_{\text{kin}}/M)_{\text{sol}}$ and $(E_{\text{kin}}/M)_{\text{halo}}$ are time-dependent as the system evolves, with the soliton slowly accreting mass, whereas the halo, after it forms, does not undergo significant evolution (see e.g. bottom-right panel of Fig. 6.10). The error bars on data points represent the change during this evolution as well as the effects of soliton fluctuations, with central values corresponding to the mean values over the evolution (see Fig. 6.10). [The maximum values of $(E_{\text{kin}}/M)_{\text{sol}}$, $(E_{\text{kin}}/M)_{\text{halo}}$ are therefore (weakly) sensitive to the simulation run time.] Horizontal error bars correspond to the fluctuations in E_{kin} towards the end of simulations.

6.2.2 Initial conditions

We study the soliton-halo configurations arising from different sets of initial conditions:

- *Soliton merging.* We start with $N = 4 - 100$ solitons randomly placed and at rest, either with (I) the *same* radius chosen in the range $L/(10 - 50)$, as in Ref. [Sch+14], or (II) with a *random* radius sampled from a uniform distribution in the range $L/(20 - 100)$, which resembles the setup of Ref. [Moc+17].
- *Halo.* We initialize the ULDM field ψ with a density approximating that of a virialized (statistically stationary) NFW [NFW97] or Burkert [Bur15] halo profile, using the Eddington procedure Eq. (5.25), Eq. (5.26). The parameters of the halos are chosen to mimic dwarf galaxy systems such as Fornax [WMO09; Wal+09] for $m = (10^{-22} - 10^{-21})$ eV, with initial density parameter $\rho_s = (10^8 - 10^{10}) \text{ M}_{\odot}/\text{kpc}^3$ and scale radius $R_s = (0.5 - 2) \text{ kpc}$.
- *Background.* We initialize ψ as a Gaussian noise background with mean density $\bar{\rho}$ and Maxwellian velocity distribution, with momentum variance $k_{\text{bkg}}^2/2$, as in Refs. [LPT18; CSX24], i.e.

$$\langle \psi^*(x) \psi(y) \rangle = C(x - y), \quad C(x) = \bar{\rho} e^{-x^2 k_{\text{bkg}}^2/4}. \quad (6.19)$$

We consider a sufficiently dense background such that the Jeans length, $\lambda_J \equiv 2\pi/(16\pi G \bar{\rho} m^2)^{1/4}$, is smaller than L , ensuring that it rapidly forms a gravitationally bound halo via the Jeans instability [LPT18]. The range of parameters probed is $k_{\text{bkg}} L = 10 - 100$ and $\bar{\rho}$ such that $\lambda_J/L = 0.1 - 0.4$.

- *Collapse.* We begin with an elliptical overdensity of the form

$$\psi(x) = \sqrt{\rho_0} e^{-x_1^2/2R_1^2 - x_2^2/2R_2^2 - x_3^2/2R_3^2}, \quad (6.20)$$

with density and radius varying in the range $\rho_0 = (0.01 - 1) \text{ M}_\odot/\text{pc}^3$ and $R_i = (0.5 - 5) \text{ kpc}$. The phase of the field is coherent across the initial configuration, but the overdensity quickly collapses and in the process loses coherence.

- *Background+Collapse.* We consider the elliptical overdensity in Eq. (6.20) on top of the noise background in Eq. (6.19).

Further details can be found in App. A.6.

6.2.3 When should a soliton-halo relation be computed?

Our goal is to clarify the properties of systems in which a soliton is embedded in a host halo whose mass is much larger than that of the soliton, $M \gg M_{\text{sol}}$. Such hierarchical systems enable kinematic tracer observables (stellar or gas velocities) in the halo's large-scale region to be compared with ULDM's velocity predictions for the inner region, where the soliton lives, enabling the soliton to be bounded or potentially discovered [Bar+18; TCB25].

To this aim, we need to choose a time at which to determine whether a soliton has formed, and, if it has, its properties and soliton-halo relation. The obvious phenomenologically-motivated choice is the Hubble time. However, the timescales involved in simulations with some types of initial conditions (e.g., those starting from a background, or soliton mergers) do not necessarily coincide with those of realistic cosmological initial conditions; in other words, the properties of the soliton and halo at one Hubble time in such simulations might not be relevant to cosmology. Nevertheless, we believe that such systems can still elucidate the dynamics leading to physical soliton-halo relation.

We therefore adopt the following procedure to determine a time, t_{form} , from which point on we analyze the soliton and halo, with the understanding that this inevitably leads to some arbitrariness. At each timestep in a simulation we identify the point in the box that contains the highest density, ρ_0 . We then determine whether the spherically-averaged density profile around this point is well-matched by the soliton solution with central density ρ_0 and half-density radius $r_c(\rho_0) = 1.3/(4\pi G m^2 \rho_0)^{1/4}$ for $r < 3r_c$ and an NFW profile for $r > 4r_c$ (with scale radius and density as free parameters). If such a fit is sufficiently good, we consider the soliton-halo system as formed. Further details and analysis of the uncertainties arising from our choices can be found in Sec. 6.3.5, while in Appendix A.7 we discuss an alternative way to pinpoint soliton formation, via the spectral energy density.

Fig. 6.10 shows results from a sample simulation with *Soliton merging (I)* initial conditions, which illustrates the formation of the soliton-halo system and motivates our procedure for when this is first identified. The top four panels show column density snapshots at $t = 0, 0.5, 1.0, 4.5 \text{ Gyr}$. The bottom-left panel shows the time evolution of M_{sol}/M , with M_{sol} estimated from the peak density ρ_0 using the soliton relation $M_{\text{sol}}^4 = 64\pi\rho_0/(G^3 m^6)$. The bottom-right panel shows the radially-averaged density profile at the corresponding times, together with the best-fit curves for the soliton and the NFW halo. Based on the criterion above, the soliton-halo system forms at $t_{\text{form}} \approx 1 \text{ Gyr}$ (dashed line). Note that if we omit the halo region ($r > 4r_c$) from the soliton-halo condition, we find that a soliton-like overdensity appears already at an earlier time with $t_{\text{form}} \approx 0.5 \text{ Gyr}$. However, the top-right panel of Fig. 6.10 shows that the halo is still evolving fast at that time. Our choice of procedure, including the choice of numerical factors, detailed in App. 6.3.5, is guided by qualitative tests of this kind. The mass of the soliton is then recovered via the fitted density profile.⁴

⁴Ref. [CSX24] pointed out that fitting the density profile with the soliton form can cause an upward bias in the inferred mass of the soliton component of the field, due to interference with unbound traveling wave fluctuations. Simple estimates

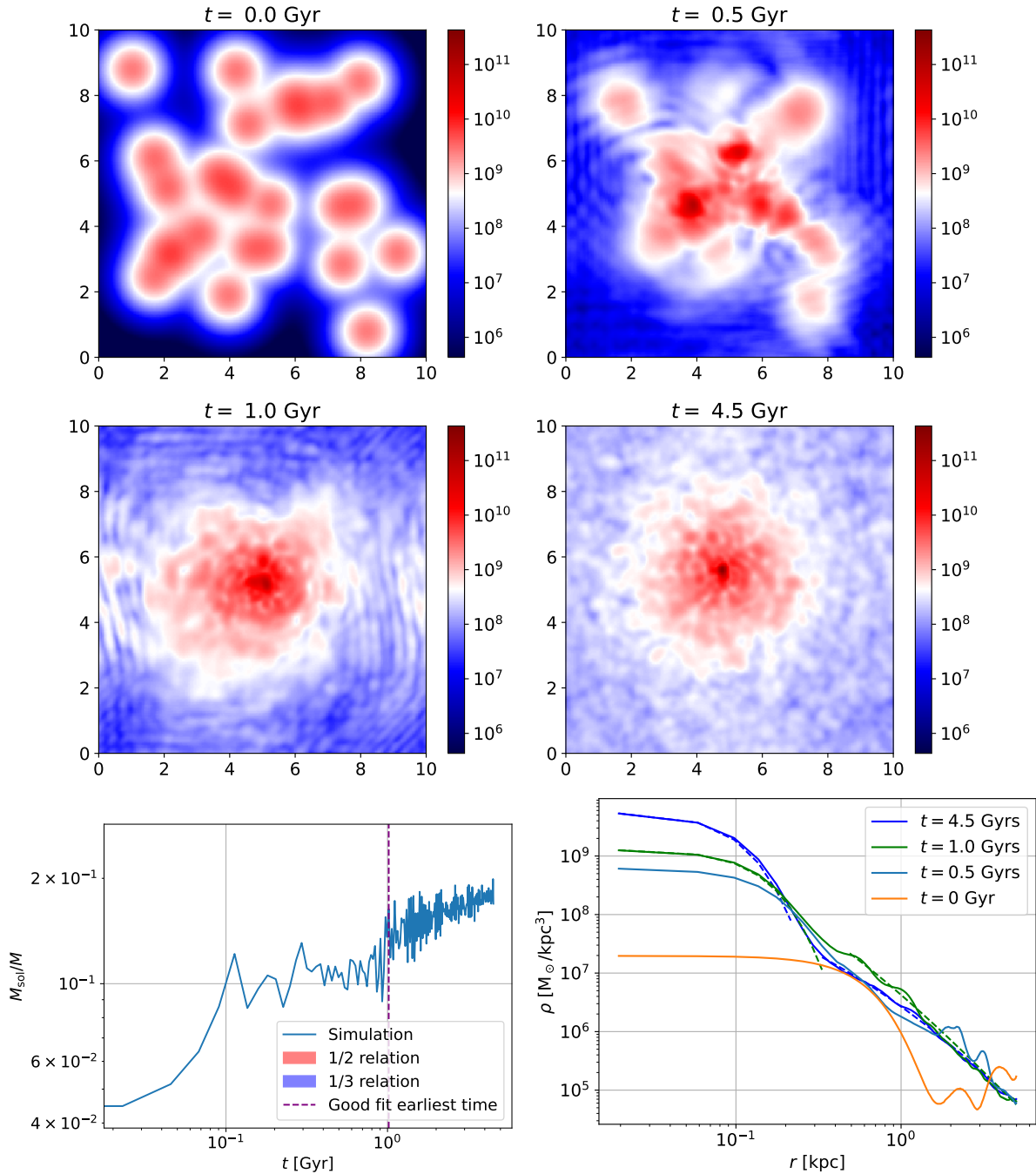


FIGURE 6.10: Results from a simulation with the *soliton merger (I)* initial conditions (in which the initial solitons have the same radius). The top four panels show the projected density field at $t = 0, 0.5, 1.0, 4.5$ Gyr, in $\text{M}_\odot/\text{kpc}^2$ units (logarithmic scale). The bottom-left panel shows the time-evolution of M_{sol}/M . We find that the virialized soliton-halo system forms at $t_{\text{form}} \approx 1$ Gyr (dashed vertical line). Red and blue bands indicate the $1/2$ and $1/3$ relations, respectively. The width of these bands reflects the short-time fluctuations in E_{kin} around the final time (corresponding to the horizontal error bars in Fig. 6.9), which affects the value of E_{kin}/M^3 and hence the prediction of M_{sol}/M . The bottom-right panel shows the radially averaged density profile around the center of the soliton at different times. The dashed lines show the soliton + NFW fit of the density profile, for the times $t = 1.0, 4.5$ Gyr at which this is good.

suggest that this effect can range between ten percent to about a factor of two. We do not aim at a better accuracy, nor believe this is required for current phenomenological applications.

Solitons exhibit quasi-normal mode fluctuations [GUL04], along with secular mass accretion from the surrounding halo [LPT18; EN19; Che+21]; both of which are apparent in the bottom-left panel of Fig. 6.10. Because of these effects, any single numerical value for M_{sol} would only capture part of the physics. Instead, when analyzing results, we report a range of M_{sol}/M values for each simulation, from $t = t_{\text{form}}$ to the final simulation time t_{final} , see e.g. Fig. 6.9. E_{kin} also undergoes fluctuations, which we represent as horizontal error bars on E_{kin}/M , which we extract from the range of values around the final time.

We end simulations when the soliton growth is deep in the era of secular growth, such that the largest value of M_{sol} reached is not very sensitive to the (somewhat arbitrary) value of t_{final} . This is due to the slow mass accretion rate during this era, and also because the growth of the soliton in finite-box simulations is bounded above, in a sense that we explain further below. The specific value of t_{final} chosen depends on the particular initial conditions. For example, for the background initial conditions we ensure that $t_{\text{final}} \gg \tau_{\text{rel}}$, where τ_{rel} is the relaxation time (see Ref. [LPT18]); for soliton merging runs, collapse etc. we ensure that t_{final} is much greater than the free-fall time t_{ff} , estimated as $t_{\text{ff}} \sim 0.5(GM/L^3)^{-1/2}$.

6.2.4 Soliton-halo relation: results

In Fig. 6.9, we summarize the soliton-halo relation obtained from all simulations and initial conditions. We express this in terms of the energy-to-mass ratios, E/M , for both the soliton and the halo. For a realistic system, the halo may extend beyond the volume that we simulate; however, as discussed below, E/M is relatively unaffected by the cutoff set by the box. To avoid potentially unphysical results from the undetermined constant in the potential energy, we use the kinetic energy E_{kin} , instead of the total energy E_{tot} .⁵ This is justified under the assumption that the system is virialized. The soliton mass M_{sol} and kinetic energy $E_{\text{kin,sol}}$ are obtained from fitting its density profile. Finally, the red line in Fig. 6.9 shows the kinetic energy version of the 1/2 relation, Eq. (6.16). Over the time-range between t_{form} and t_{final} , the data points fall around the 1/2 relation.

To compare with existing literature and other soliton-halo relations, in particular the 1/3 relation, Eq. (6.17), we follow Refs. [Moc+17; Zag+23] in defining the dimensionless variable

$$\Xi \equiv \frac{|E_{\text{tot}}|}{M^3} \frac{1}{G^2 m^2}. \quad (6.21)$$

This is an invariant of the Schrödinger–Poisson system under the residual rescaling symmetry described in App. A.1 and is thus a natural candidate to enter in a soliton-halo relation. A class of soliton-halo relations, including the 1/2 and 1/3 relations, can be parametrized as

$$\frac{M_{\text{sol}}}{M} = \alpha \Xi^\beta. \quad (6.22)$$

In the literature, α and β have been fitted as independent parameters. However, requiring the relation to hold in the limit $M_{\text{sol}} \rightarrow M$ fixes $\alpha \simeq (0.054)^{-\beta}$ (see App. A.2). Perhaps not surprisingly, given that the fits to simulation data were obtained from systems with M_{sol} not much smaller than M , previously reported numerical values of α and β are consistent with this limit. For example, the $\beta = 1/2$ relation in Ref. [Sch+14] gives $\alpha \approx 4.2$, while Ref. [Moc+17] finds $\beta = 1/3$ and $\alpha \approx 2.6$.

In Fig. 6.11, we present the same results as in Fig. 6.9, but using the variables M_{sol}/M and Ξ . As before, we replace $|E_{\text{tot}}|$ in Eq. (6.21) by E_{kin} . Consistently with Fig. 6.9, our results for M_{sol}/M fall around the 1/2 relation. They also lie below the 1/3 relation, with the trend becoming clearer at small M_{sol}/M . This agrees with our expectation that the 1/3 relation is an upper bound. To illustrate this point further,

⁵See discussion in Section IIA of [Bar+19]. Given this undetermined constant, plotting results in terms of $E_{\text{tot,(sol)}}$ may introduce confusion when comparing different groups.

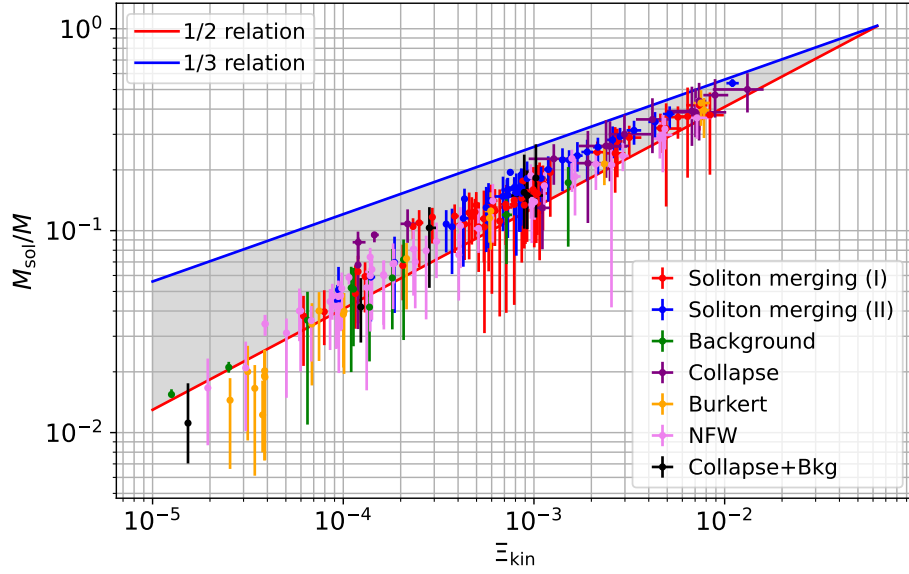


FIGURE 6.11: The ratio M_{sol}/M between the soliton and halo masses obtained in numerical simulations, as a function of the dimensionless Schrödinger–Poisson invariant $\Xi = |E|/(M^3 G^2 m^2)$, where E is calculated from the kinetic energy using the virial theorem. The red and blue lines show the 1/2 and 1/3 relations, respectively. Once the soliton-halo system is formed, it typically lies within (or at most slightly below) the band bracketed by these two relations, shaded gray. Error bars reflect variations in the soliton-halo relation over the system’s evolution (as M_{sol}/M increases) and the impact of soliton fluctuations.

in Fig. 6.12 we plot $E_{\text{kin,sol}}$ versus the total kinetic energy E_{kin} , i.e. the kinetic version of Eq. (6.18), confirming $E_{\text{kin,sol}} < E_{\text{kin}}$.

Despite its convenience in formulating the soliton-halo relation and describing simulation results, Ξ is unfortunately badly affected by finite-volume effects. On the other hand, the quantity E/M used in the relation in Eq. (6.16) is much less sensitive to such artifacts. To illustrate this, in Fig. 6.13 we show the r -dependence of $|E(r)|/M(r)$ and $\Xi(r) \propto |E(r)|/M^3(r)$ for an NFW halo of scale radius R_s by integrating E and M separately from the halo center to the point r . This procedure mimics how such variables vary as a function of the box size in finite-volume simulations that do not extend to the virial radius of a simulated halo system. $E(r)/M(r)$ only changes by a factor of 2 as r varies from R_s to $10R_s$, however $\Xi(r)$ varies by more than two orders of magnitude.

As further illustration of the weak dependence of E/M on the box length L , in Fig. 6.14 we show results from three simulations for $m = 10^{-21}$ eV starting from NFW halo initial conditions with same central density and scale radius $R_s = 0.5$ kpc, but different $L = 5, 10, 20$ kpc. A prominent soliton forms, with its mass varying by at most 10% across the three box sizes. The left and center panels compare the soliton-halo relations using E/M and Ξ , respectively. E_{kin}/M and $E_{\text{sol,kin}}/M_{\text{sol}}$ remain clustered within 20% across the three cases, whereas Ξ and M_{sol}/M vary by a factor of 2 and 5, and thus are much more sensitive to L .

Importantly for observational comparisons, a soliton-halo system that lies around the 1/2 relation leads to a velocity rotation curve $v_{\text{circ}}(r) := \sqrt{GM(r)/r}$ that has two distinct and approximately equal peaks, at distances of the order of the soliton radius and the halo scale radius, respectively [Bar+18]. The soliton peak predicted from simulations is largely unaffected by finite-volume effects. Indeed, $v_{\text{circ}}(r)$ only depends on the mass distribution at a distance smaller than r (for an approximately spherical system), and is thus sensitive to finite box size effects only to the extent that these alter the density distribution of the system within the simulated volume. To demonstrate this, in the right panel of Fig. 6.14 we plot

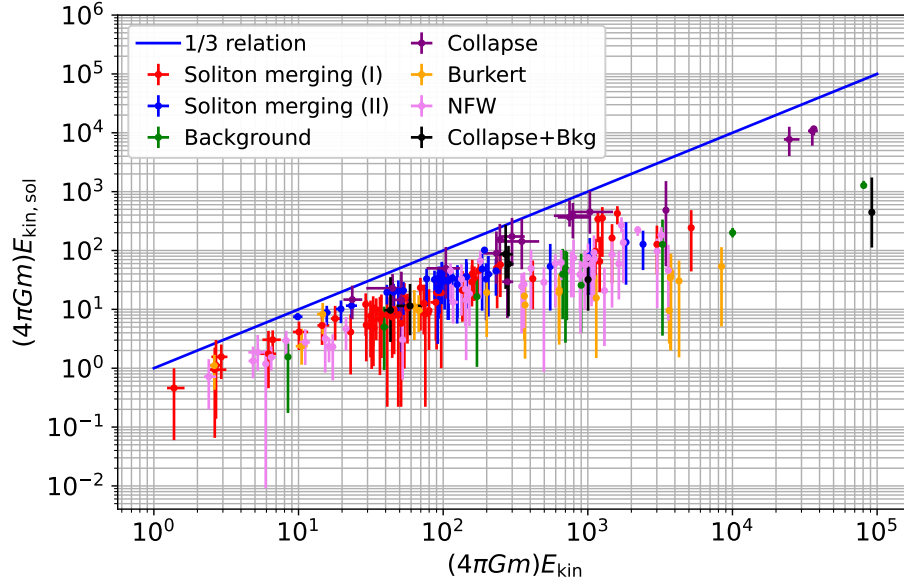


FIGURE 6.12: The soliton kinetic energy, $E_{\text{kin,sol}}$, compared to the total kinetic energy in the simulation box, E_{kin} (which approximates the halo's plus soliton's kinetic energy). We also show the line $E_{\text{kin,sol}} = E_{\text{kin}}$. As expected, $E_{\text{kin,sol}} < E_{\text{kin}}$ in all cases, which is consistent with all data points falling below the 1/3 soliton-halo relation in Fig. 6.11.

the rotation curves in the three simulations in different size boxes described above. The amplitude and position of the inner peak, associated to the soliton, is the same within 20% in all cases. The outer peak, linked to the halo, is instead sensitive to box size if this is too small; for instance, it is absent in a 5 kpc box.

6.3 Discussion of our soliton-halo simulations results

6.3.1 Consistency with the evaporation lower limit of Chan et al. 2022

Chan, Sibiryakov, and Xue [CSX24] conducted a perturbative analysis of soliton mass absorption and evaporation in a statistically-homogeneous background with a Maxwellian velocity distribution and characteristic momentum k_{bkg} , as given by Eq. (6.19). They demonstrated that a soliton undergoes evaporation due to gravitational scattering when the background particles are significantly faster than those in the soliton. More precisely, this occurs if⁶

$$\frac{E_{\text{kin,sol}}}{M_{\text{sol}}} \lesssim 0.12 \frac{k_{\text{bkg}}^2}{2m^2}. \quad (6.23)$$

In the opposite limit, the soliton instead grows by accreting mass from the background. The 1/2 relation of Eq. (6.16), phrased in terms of kinetic rather than total energy, can be written in a form similar to Eq. (6.23):

$$\frac{E_{\text{kin,sol}}}{M_{\text{sol}}} = \frac{E_{\text{kin}}}{M} = \frac{3k_{\text{bkg}}^2}{4m^2}, \quad (6.24)$$

where we used $E_{\text{kin}}/M = 3k_{\text{bkg}}^2/4m^2$, valid for a Maxwellian distribution with variance $k_{\text{bkg}}^2/2$, see App. A.6. Comparing Eq. (6.24) with Eq. (6.23), we obtain that solitons that just exceed the evaporation limit of Eq. (6.23) are a factor $(3/(2 \cdot 0.12))^{1/2} \sim 3.5$ lighter than prescribed by the 1/2 relation, i.e.

⁶This is obtained combining Eqs. (2.18) and (2.12) of Ref. [CSX24] with the condition $T_s/T_g \lesssim 0.08$ obtained in the same reference.

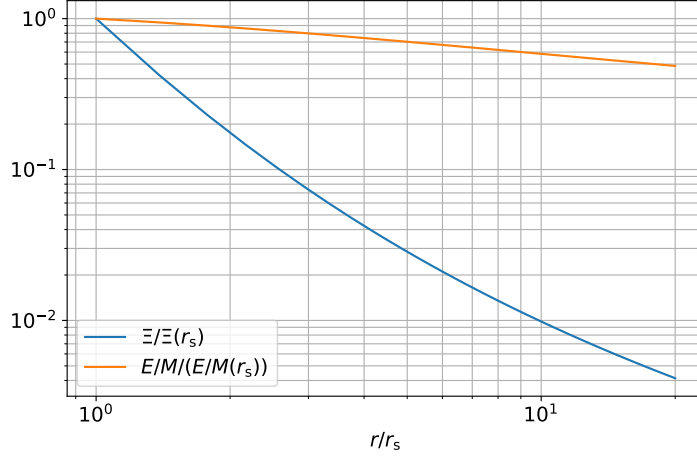


FIGURE 6.13: The radial dependence of the Schrödinger–Poisson invariant $\Xi(r) \propto |E(r)|/M(r)^3$ and of $|E(r)|/M(r)$ in an NFW halo with scale radius R_s , normalized to their values at $r = R_s$. These are calculated using the analytic form of the NFW density profile and gravitational potential. $\Xi(r)$ has a much stronger dependence on r than $|E(r)|/M(r)$, and therefore when evaluated in simulations is more sensitive to the finite-box size in which a soliton-halo system is contained.

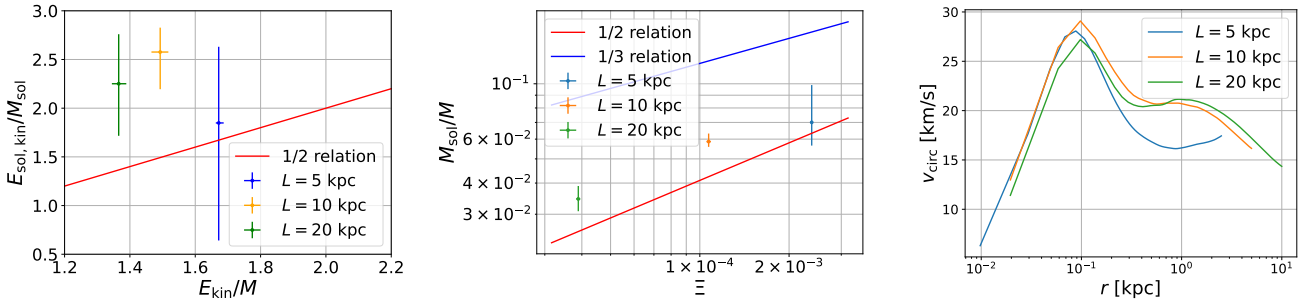


FIGURE 6.14: The soliton-halo relation expressed in terms of E/M (left panel) and Ξ (center panel), arising from simulations with the same halo initial conditions but in different sized boxes, with box length L . The soliton-halo relation shows significantly less sensitivity to L when expressed in terms of E/M rather than Ξ . We also show the circular velocity $v_{\text{circ}}(r)$ in the right panel.

solitons evaporate if their mass is smaller than

$$M_{\text{sol, evap}} \simeq 1.2 \left(\frac{|E_{\text{tot}}|}{M} \right)^{1/2} \frac{1}{Gm}. \quad (6.25)$$

Although strictly valid only for a soliton embedded in a Maxwellian gas, Eqs. (6.24) and (6.25) may still be applicable to a generic soliton-halo system, where E_{kin} , M , and k_{bkg} are interpreted as the kinetic energy, mass, and characteristic momentum in an approximately virialized region that is larger than the soliton itself and containing a mass much larger than the soliton mass. In this case, only solitons with masses at most a factor of ~ 3.5 below the 1/2 line (red) in Fig. 6.11 would be possible. The inhomogeneous nature of a halo and the fact that the particles in the halo are self-bound could affect this analysis. However, the evaporation limit of Eq. (6.25) appears to nevertheless be approximately valid, see Fig. 6.11.

In particular, simulations are roughly consistent with such a limit. An example of this behaviour is plotted in Fig. 6.15, which shows the growth of a soliton starting from background noise initial conditions

(with *initial* $k_{\text{bkg}}L = 40$, $\lambda_J/L \simeq 0.2$). Such initial conditions are fairly close to the simplifying assumptions in the analysis of Ref. [CSX24], except for the fact that $\lambda_J/L < 1$, which induces background collapse via Jeans instability and the formation of a self-bound halo. As shown in the bottom-left panel of Fig. 6.15, when it forms, the soliton lies between the 1/2 relation in Eq. (6.15) (red) and the evaporation limit in Eq. (6.25) (dashed blue), both computed using E_{kin} from the simulation after soliton formation. Note that the evaporation limit is calculated for the virialized halo, i.e. after the collapse of the initial background waves. During collapse E_{kin} becomes significantly larger than its initial value $3k_{\text{bkg}}^2 M/4m^2$, leading to a tighter evaporation limit than that one would infer from the initial kinetic energy density.

6.3.2 The 1/3 relation is an upper bound

Since the 1/3 relation simply amounts to Eq. (6.18), it can at most set an upper bound on the soliton mass in a cosmological context, or in flat-space simulations where all field components are bound (i.e., have negative energy). In a finite volume, there could be unbound positive-energy debris that orbit the box, which would allow the 1/3 relation to be violated if E_{tot} and M are interpreted as the halo's energy and mass (in an infinite volume, such debris would escape to infinity). However, by analyzing the spectrum of the ULDM field, see App. A.7, we find that our simulations have a negligible quantity of positive energy debris. Figs. 6.11 and 6.12 are consistent with this point and thus the 1/3 relation. Our presentation of kinetic rather than total energy in these figures avoids ambiguities with respect to the definition of total energy that would affect the soliton-halo relation; these however need to be resolved for direct comparison to other works in the literature, notably Ref. [Moc+17], which makes use of the total energy. We return to this point in Sec. 6.3.4.

6.3.3 Observational implications

Given sufficient time, halos in our simulations grow a central soliton with an initial mass lying near the 1/2 relation that then slowly increases in mass towards the 1/3 relation. As we discuss further in Section 6.3.4, we expect similar scaling for cosmological halos. A soliton-halo relation allows predictions to be made for observable rotation and dispersion velocity curves [Bar+18]. A soliton of a given mass M_{sol} induces a characteristic peak in the circular velocity profile at radius r_{peak} with value v_{peak} , which are given by

$$v_{\text{peak}} \approx 8.3 \left(\frac{m}{10^{-21} \text{eV}} \right) \left(\frac{M_{\text{sol}}}{10^7 M_{\odot}} \right) \text{km s}^{-1}, \quad r_{\text{peak}} \approx 0.46 \left(\frac{m}{10^{-21} \text{eV}} \right)^{-2} \left(\frac{M_{\text{sol}}}{10^7 M_{\odot}} \right)^{-1} \text{kpc}. \quad (6.26)$$

As mentioned, for a density profile consisting of a soliton embedded in a host halo and not too far from the 1/2 relation, a doubly-peaked rotation curve is predicted. The inner peak is approximately given by Eq. (6.26), and is at radius $r \sim 2r_c$ [Bar+18], and the outer peak is induced by the NFW halo at radius $r \sim 2R_s$. As the mass of a soliton increases above the 1/2 relation the relative amplitude of the second peak decreases. If the radial scale $r \sim 2r_c$ can be resolved by observations (recall that $r_c \propto 1/m$), the presence or lack thereof of the inner peak can constrain, or give evidence for, ULDM with a particular m .

Fig. 6.16 shows the circular velocity curves of soliton-halo systems in simulations with halo initial conditions (NFW and Burkert). These are calculated from the radially-averaged density profiles, and normalized to v_{peak} and r_{peak} , which are defined in Eq. (6.26) and evaluated using the soliton density profile fit. These results are plotted at a physical time of 10 Gyr, and the masses and size of the halos are taken in the range relevant to dwarf galaxies and ULDM particle mass m between 5×10^{-22} eV and 10^{-21} eV. In particular, we vary the box length between 10 kpc and 40 kpc, and the halo parameters within $\rho_s = 10^6 - 10^8 M_{\odot}/\text{kpc}^3$, $R_s = 0.5 - 1.5$ kpc.

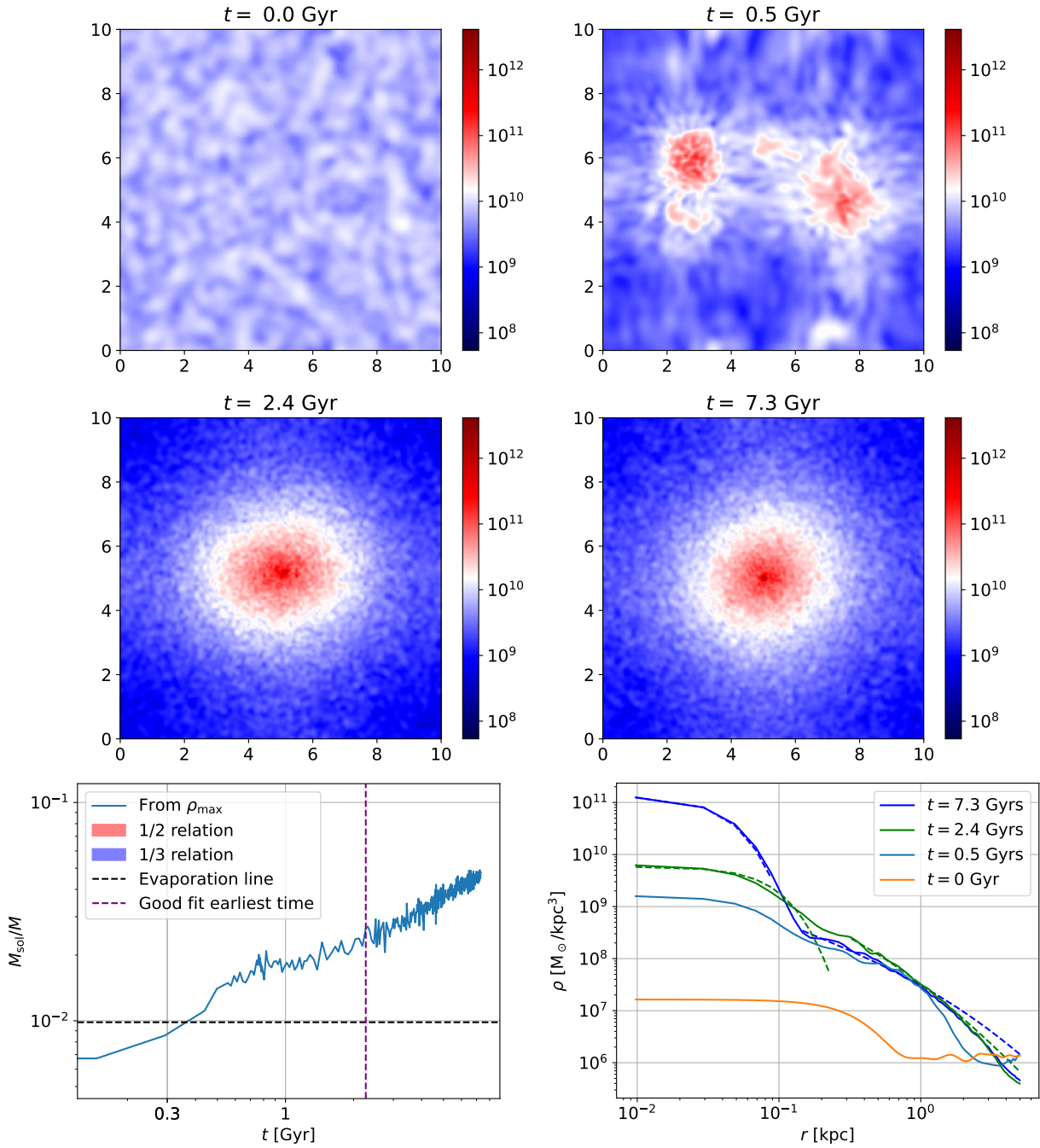


FIGURE 6.15: Formation the soliton-halo system and growth of the soliton in a typical simulation starting from initial conditions consisting of background noise, with Jeans wavelength $\lambda_J/L \simeq 0.2$. The plots are as described in Fig. 6.10; in the bottom-left panel we additionally show the evaporation limit of Ref. [CSX24], below which a soliton is unstable to evaporation by scattering with the background. This limit is a factor of roughly 3.5 smaller the 1/2 relation (in this plot the finite thickness of the 1/2 and 1/3 bands is hardly noticeable).

Doubly-peaked rotation curves are generally *not* observed in the kinematical data of DM-dominated disk galaxies. This led Refs. [Bar+18; Bar+19; BBS22] to infer a bound $m \gtrsim 10^{-21}$ eV by assuming the 1/2 soliton-halo relation. As this relation is consistent with our results for $m \simeq 10^{-21}$ eV in dwarf galaxy-like systems with ages near the Hubble time, our simulations support such a lower bound on m .

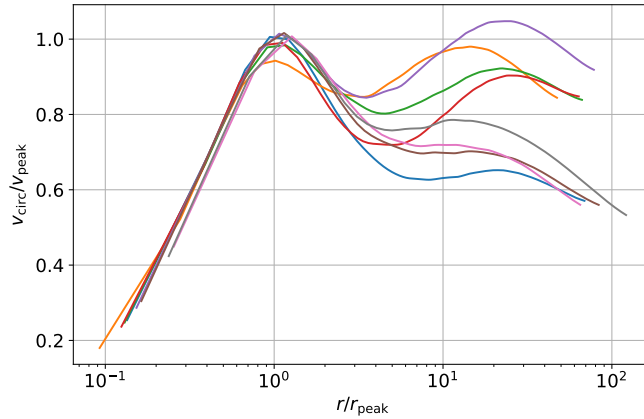


FIGURE 6.16: Velocity rotation curves for simulations with halo initial condition (Burkert and NFW), evaluated at $t \approx 10$ Gyr. Here, r_{peak} is the distance at which the circular velocity v_{circ} reaches its maximum for a soliton density profile matched to the central density in the simulations, and v_{peak} is the corresponding velocity. The rotation curves plotted are obtained from the radially averaged density profile in the simulations (and hence v_{circ} is not exactly equal to v_{peak} at $r = r_{\text{peak}}$).

6.3.4 Comparison with previous works

Comparison with flat space-time simulations

We now compare with Ref. [Moc+17] and examine the consistency of our results with their claimed $1/3$ relation. Our numerical methods (i.e. our algorithm to solve the SPE) and the setting (flat space-time with periodic boundary) are similar to this reference. Moreover, as far as we could tell, our *soliton merging (II)* initial conditions match their initial conditions. Yet our results, as shown in Fig. 6.11, lie systematically below the $1/3$ relation, in apparent contrast to those of [Moc+17].

One possible reason for this discrepancy is the choice of Ref. [Moc+17] to use the variable Ξ calculated from the total energy E_{tot} rather than the kinetic energy. Since, as mentioned, in a finite volume the potential energy E_{pot} contains an arbitrary additive constant, the total energy does too. If not chosen such that the potential energy matches its infinite volume value, such a constant means that e.g. the system would usually not obey the virial theorem $E_{\text{kin}} = -E_{\text{tot}}$.⁷

To check that this is the origin of our discrepancy with Ref. [Moc+17], in the left panel of Fig. 6.17 we plot our results in terms of Ξ calculated using the total energy without any correction in E_{pot} . With this prescription our results cluster around the $1/3$ relation, slightly exceeding it in some cases. Meanwhile, in the right panel we plot only data from *soliton merger (II)* initial conditions (blue markers), which are the direct parallel of the simulations of Ref. [Moc+17]. The results obtained in this way are fully consistent with those of [Moc+17]; for comparison, we show also digitized data from Fig. 4 there (black points).

Other works in the literature used different simulation setups to analyze the soliton-halo relation, so only indirect comparisons are possible. Ref. [Zag+23] used absorbing boundary conditions. Since, in this case, Ξ is not constant in time, Ref. [Zag+23] presented results both with respect to the initial value of Ξ , and with respect to a short time average $\langle \Xi \rangle$. In Fig. 6.18 we superimpose the $1/2$ and $1/3$ relations alongside digitized results from that paper. We do the same for the results in Ref. [SNE16], which also used absorbing boundary conditions, again in Fig. 6.18.⁸ Speculatively, if these references did use

⁷Fig. 1 in Ref. [Moc+17] shows the total, potential, and kinetic energy in their simulation. That plot shows that the virial theorem is indeed violated at the $\mathcal{O}(1)$ level. Digitizing the plot we find $2E_{\text{kin}}/|E_{\text{pot}}| \approx 1.4$, instead of 1.

⁸We could not determine whether Ξ used in that reference was calculated from the initial data, or another procedure as in [Zag+23].

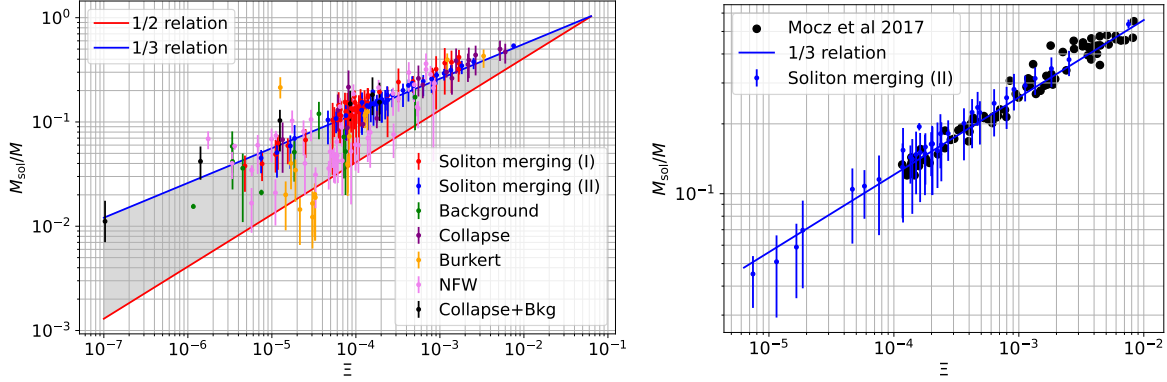


FIGURE 6.17: Left: the same as Fig. 6.11, but with Ξ calculated using the total (instead of kinetic) energy. In the total energy, the potential energy is evaluated directly in finite volume, i.e. without choosing the arbitrary constant to match the infinite-volume limit. Right: the same as the left plot, but for direct comparison with Ref. [Moc+17], we show only our results for *soliton merger (II)* initial conditions (blue markers), as used in Ref. [Moc+17]. Digitized results from Ref. [Moc+17] are shown by black dots.

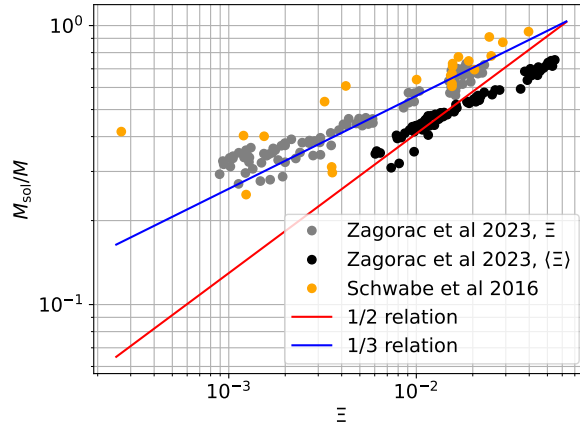


FIGURE 6.18: Simulation results for the soliton halo relation from Ref. [Zag+23] and Ref. [SNE16], compared with the 1/2 and 1/3 relations.

an unadjusted E_{pot} , that might (possibly partially) account for those data points clustered around and above the 1/3 line. A direct comparison by re-analyzing such data using E_{kin} may be useful.

Comparison with cosmological simulations

In a cosmological context, the soliton-halo relation is usually written as

$$a^{1/2} M_{\text{sol}} = A_h \left(\sqrt{\frac{\zeta(a)}{\zeta(1)}} M \right)^{B_h}, \quad (6.27)$$

where M_{sol} and M are the mass of a soliton and halo respectively, and A_h and B_h are parameters of the relation. Here, a is the cosmological scale factor, and $\zeta(a)$ is the cosmology-dependent factor that relates the average overdensity of the halo within its virial radius, r_{vir} , to the cosmological dark matter density at the time of formation [Sch+14], i.e.

$$M \equiv \frac{4\pi}{3} \frac{\rho_m}{a^3} r_{\text{vir}}^3 \zeta(a), \quad (6.28)$$

and ρ_m is the cosmological dark matter density today. Note that $\zeta(a) = (18\pi^2 + 82(\Omega_m(a) - 1) - 39(\Omega_m(a) - 1)^2)/\Omega_m(a)$, with $\Omega_m(a)$ the matter cosmological abundance at scale factor a [BN98].

We can attempt to match the general form of the soliton-halo relations we consider, Eq. (6.22), with the cosmological form of Eq. (6.27) as follows. For a virialized halo, it is reasonable to assume

$$|E_{\text{tot}}| \sim \frac{GM^2}{r_{\text{vir}}}. \quad (6.29)$$

Plugging this into Eq. (6.22) with r_{vir} expressed in terms of M from Eq. (6.28), leads to

$$a^\beta M_{\text{sol}} = \alpha_\beta \left(\frac{4\pi\rho_m\zeta(1)}{3G^3m^6} \right)^{\frac{\beta}{3}} \left(\frac{\zeta(a)}{\zeta(1)} \right)^{\beta-\frac{1}{2}} \left(\sqrt{\frac{\zeta(a)}{\zeta(1)}} M \right)^{1-\frac{4\beta}{3}} \implies B_h(\beta) = 1 - \frac{4\beta}{3}. \quad (6.30)$$

We see that this only leads to a soliton-halo relation of the form of Eq. (6.27) for $\beta = 1/2$, in which case $B_h = 1/3$ (this was also found in Ref. [Sch+14]). On the other hand, for a generic β , the most we can do is neglect the effects of cosmology, by setting $a = 1$, $\zeta = 1$. With such an assumption, we can relate the parameters A_h for different β as

$$A_h(\beta) = \alpha_\beta \left(\frac{A_h(\beta')}{\alpha_{\beta'}} \right)^{\beta/\beta'} \implies A_h(1/3) \approx A_h^{2/3}(1/2), \quad (6.31)$$

where we used $\alpha_{1/2} = 4.2$ and $\alpha_{1/3} = 2.6$.⁹

Fig. 6.19 shows results from cosmological simulations [Sch+14; MS21; NB21; CSX24] as collected by Ref. [Cha+22], together with some halos from Ref. [MMW22]. In Ref. [MMW22], results from 5 halos at different redshift are shown. In Fig. 6.19, we only show results from the halos at their latest redshift $z = 2.5$ and exclude their halo-4; as one can evince from Fig. 4, Fig. 6 *there*, such excluded halos may not be virialized. Ref. [MMW22] used $m = 2.5 \times 10^{-22}$ eV, whereas Ref. [Cha+22] used $m = 0.8 \times 10^{-22}$ eV. According to the rescaling Eq. (A.3), Eq. (A.9), to translate the points of Ref. [MMW22] to $m = 0.8 \times 10^{-22}$ eV while keeping the same time-scales, one should rescale $M \rightarrow (2.5/0.8)^{3/2}M$. We compare these with the $1/2$ cosmological relation of Eq. (6.30) with $\beta = 1/2$, and also with the $1/3$ relation in the form of Eq. (6.27) neglecting cosmological effects (i.e. with $a = 1$ and $B_h(1/3)$ and $A_h(1/3)$ as in Eqs. (6.30) and (6.31)). We find rough agreement with the expectations that the $1/2$ relation is a rough lower bound and the $1/3$ relation is an upper bound, despite neglecting cosmological effects in the latter¹⁰.

6.3.5 The criterion for soliton formation

Here, we provide details of the criterion used to define the formation time of a soliton-halo system, as briefly discussed in Section 6.2.3. Explicitly:

1. At each time step in the simulation we identify the in the box that contains the highest mass density. Call this point the origin of coordinates $r = 0$, with density $\rho(0)$. Using the soliton profile approximation [SCB14]

$$\rho_{\text{fit}}(r) = \frac{\lambda_s^4}{(1 + a^2\lambda_s^2r^2)^b}, \quad a \approx 0.228, \quad b \approx 4.071, \quad (6.32)$$

⁹To analyze such data without neglecting cosmological effects, we would require soliton and halo masses without cosmological normalizations, which have typically not been reported in the literature.

¹⁰The results of Ref. [NB21] are outliers with respect to the other groups results we show. Although we do not have a clear explanation as of why this is the case, we note that Ref. [NB21] uses a smoothed particle hydrodynamic approach that emulate the ULDM wave effects, which may affect the soliton-halo relation obtained.

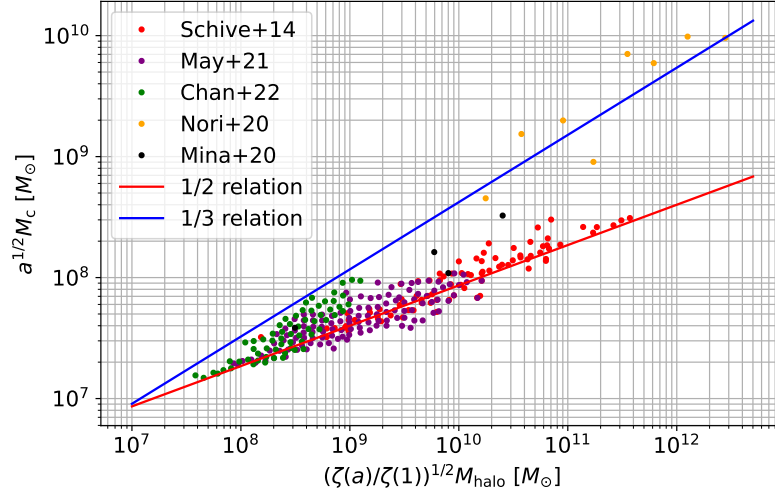


FIGURE 6.19: Results from cosmological simulations [Sch+14; MS21; NB21; CSX24], as collected by Ref. [Cha+22], plus from Ref. [MMW22], scaled for an ULDM mass $m = 8 \times 10^{-23}$ eV (results in Ref. [MMW22] are shown for $m = 2.5 \times 10^{-22}$ eV). Notice that here, results are expressed using $M_c \simeq M_{\text{sol}}/4.2$. (For Ref. [MMW22] we show only halos at $z = 2.5$, and exclude their halo-4 that seems far from virialized, see Figs. 4 and 6 *there*.) Also shown is the 1/2 relation translated to a cosmological setting, and the 1/3 relation (the latter assumes no cosmology effects, see discussion in Sec. 6.3.4).

we obtain a first estimate of the soliton core radius r_c from the value of $\rho(0)$.

2. Next, we compute the radial-averaged density profile around $r = 0$. We fit a soliton profile to the grid region surrounding $r = 0$, treating r_c as a fit parameter, and including in the fit grid points within $r < 3r_c$.

At the same time, to constrain the halo, we also fit an NFW profile to grid points satisfying $r > 4r_c$, with the NFW scale radius r_s and density ρ_s as fit parameters. Namely, we add

$$\rho_{\text{fit}}(r > 4r_c) = \frac{\rho_s}{\frac{r}{R_s} \left(1 + \frac{r}{R_s}\right)^2}. \quad (6.33)$$

3. The earliest time t_{form} that satisfies

$$\frac{1}{N} \sum_{i \in \text{fit points}}^N \log^2 \left(\frac{\rho_{\text{sim}}(r_i, t_{\text{form}})}{\rho_{\text{fit}}(r_i)} \right) < 0.5, \quad (6.34)$$

is where we begin to report results in the context of a soliton-halo relation.

As we show in Fig. 6.20, changing the threshold in Eq. (6.34) does not significantly change the results we obtain. A more restrictive criterion selects later formation times, making points in the soliton-halo diagram start their life when they are more massive. Meanwhile, a less restrictive criterion, e.g. increasing the threshold in Eq. (6.34) to 5, initially less massive solitons. However, such a criterion is actually so weak that in some runs an examination of the field and density profile suggests there is actually no soliton at some times when the threshold is met, limiting the reliability of the soliton-halo relation plot obtained.

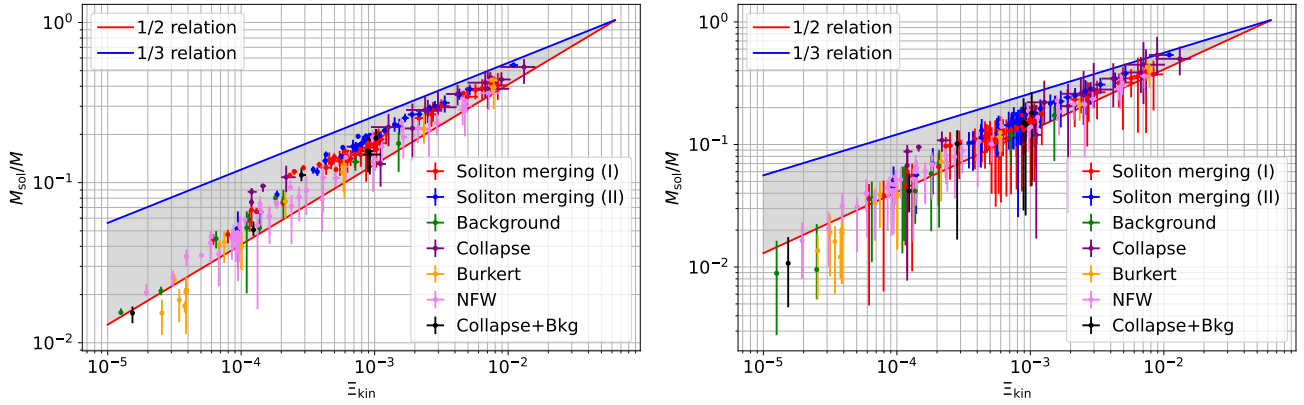


FIGURE 6.20: The same plot as in Fig. 6.11, but with different thresholds used in Eq. (6.34) for determining the moment of soliton formation: 0.05 and 5 in left and right panel respectively, instead of 0.5.

6.4 Summary of the soliton-halo relation results

We performed a series of flat-space numerical simulations of ULDM halos initialized in different ways, with the aim of analyzing on the soliton-halo relation, exploring a larger range of M_{sol}/M than previous works. Our main findings are summarized in Figs. 6.9 and 6.11. In all of our simulations, solitons form around the 1/2 relation, and then accrete mass from the surrounding halo without surpassing the 1/3 relation. This can be explained by the 1/3 relation being an upper bound, whereas the 1/2 relation scales parameterically as, and is a factor of about 3 above the evaporation lower bound derived in Ref. [CSX24].

After accounting for an ambiguity in the definition of the potential energy in a finite volume, our results are compatible with data from other works in the literature. We believe it would be useful if other groups who have carried out flat-space simulations re-analyzed their data taking into account this issue with the potential energy to check whether they also find solitons lying within the band between the 1/2 and 1/3 relations. If this is indeed the case, then the soliton-halo relation could be viewed as a “soliton-halo band” bracketed by these two relations.

We have also noted that the quantity Ξ (defined in Eq. (6.21)), which is often used in the literature to express a soliton-halo relation, is sensitive to finite-volume effects.

If solitons still fall into the soliton-halo band once cosmological effects are included (as Fig. 6.19 suggests might be the case), then the use of the 1/2 relation when investigating observational consequences can be regarded as reasonably conservative. In particular, we predict solitons to have mass roughly given by the 1/2 relation or larger and as a result the inner peak in the rotation curve, induced by the soliton, is generically of similar amplitude or larger than the outer one induced by the halo. An important remaining question is the timescale on which the soliton forms cosmologically: if the relaxation timescale is relevant then the soliton-halo predictions hold for dwarf galaxy systems (velocity dispersion of the order of 10 km/s) for ULDM masses $m \lesssim 10^{-20}$ eV.

Chapter 7

Summary

For the past decades, the scientific community has been striving to better understand the nature of dark matter beyond the paradigm of Λ CDM. Despite arising tensions, most notably the Hubble tension, these debates have not brought us significantly closer to a deeper understanding on these fundamental physics issues. However, with already operating missions such as DES [Tre+18], Euclid [Bla+20], DESI [Lev+13; Agh+16a; Agh+16b] and JWST [Gar+06], and upcoming ones like the Nancy Grace Roman Space Telescope [Spe+15], the Square Kilometer Array [Bac+20] and the Vera C. Rubin Observatory (LSST) [Abe+09], we will have access to a vast amount of new astrophysics and cosmology data, which may help us in getting closer in understanding the nature of dark matter and how our universe really behaves on large scales.

In line with this opportunity, in this thesis we tried to delve deeper into what we can achieve in the understanding of dark matter and cosmology with increasingly precise data on galaxies. On the dark matter side, we showed how in principle one can look for ULDM cores using gravitational lensing time delays, covering a mass range which is as of now not constrained by other probes. We also investigated the effect on stellar kinematics of ULDM, by looking at the dynamical heating of stars and at the soliton-halo relation via numerical simulations. New and more precise data on dwarf galaxies and ultra-faint dwarf galaxies will help us constrain the ULDM paradigm via this avenue.

On the cosmology side, we focused on the gravitational lensing time-delay probe of H_0 , addressing possible degeneracies and caveats arising from not-proper accounting of large scale structures and host group influences, and possible internal cores. The latter may indeed come from ULDM, possibly “explaining” the high H_0 value TDCOSMO I [Mil+20] found, when not explicitly considering an internal mass sheet. We also discussed under which assumptions multiple source systems can be used to break at least partially some of these degeneracies. The study of multiple source systems is timely, as forecasts predict many more multiple source systems to be discovered [Sha+24].

A full list of the main results of this thesis, based on Refs. [BT21b; Teo+22; TB23; BT25; TCB25; Blu+25] is as follows.

- Ref. [BT21b]. We consider in Ch. 5 ULDM cores as a possible source of approximate internal MSD. We show in Fig. 5.2 that cores which can bias down the H_0 measurements of TDCOSMO systems [Mil+20] back to the CMB value can arise, if we rely on relaxation time estimates Eq. (5.35). The scenario we portray needs ULDM of mass $m \sim 10^{-25}$ eV which makes up a fraction of order 10% of the total cosmological dark matter density, which is compatible with all other cosmological constraints. Stellar kinematics measurements of well-resolved massive galaxies (including the Milky Way) may offer a way to test the scenario. Kinematics analyses aiming to test the core hypothesis in massive elliptical lens galaxies should not, in general, adopt the perfect mass sheet limit, as ignoring the finite extent of an actual physical core could lead to significant systematic errors, as we show in Sec. 4.1.1.

- Ref. [Teo+22]. We consider the impact in time delay cosmography of weak lensing effects associated with different segments of the line of sight. We show in Sec 4.3.2 that external convergence terms associated with the lens-source and observer-lens segments need to be included in cosmographic modelling, in addition to the usual observer-source term, to avoid systematic bias in the inferred value of H_0 . Specifically, we show how an incomplete account of the observer-lens line of sight biases the stellar kinematics use to alleviate the MSD. Moreover, we consider the use of imaging data for multiple strongly-lensed sources in a given system. We show in Sec. 4.3.1 that the MSD is not fully resolved by the availability of multiple sources: some degeneracy remains because of differential external convergence between the different sources. Similarly, as we show in Sec. 4.3.4, differential external convergence also complicates the use of multiple sources in addressing the approximate MSD associated with an internal core component in lens galaxies, like the ULDM one discussed in Ref. [BT21b]. We illustrate this point in Fig. 4.19, where we compare the apparent MSD breaking, coming from two examples of multiple source systems, with the differential external convergence expectation between different sources. The latter may hinder the possibility to resolve an internal MSD for these systems. On the other hand, if differential external convergences are within control, then Eq. (4.65) offers a way to spot the influence of unaccounted internal cores.

For a rough assessment of the weak lensing effects, we provide estimates of external convergence using the non-linear matter power spectrum, Sec. 4.3.5, paying attention to non-equal time correlators. For the latter, we introduced a novel approximation, discussed in Sec. 1.1.6.

- Ref. [TB23]. Massive elliptical galaxies, that serve as lenses in gravitational lensing time delay measurements of the Hubble parameter H_0 , often reside in a host group. We considered degeneracies in the modelling of the group halo in Sec. 4.2, having the example of the PG1115+080 system in mind. When the group effect on imaging can be summarized by its flexion (the next order term beyond shear in the tidal expansion), the posterior likelihood map can develop disjoint local minima, as it is apparent in Fig. 4.5. This is associated with an approximate discrete symmetry of a dominant flexion term. MCMC algorithms that are not designed to explore a rich posterior landscape can miss some of the minima, introducing systematic biases. We study mock data and demonstrate that the bias in H_0 can exceed 10%, and pulls the inference value of H_0 above its truth value, for a reason that can be traced to the structure of a mis-modelled flexion term, see Sec. 4.2.4. MCMC algorithms that are designed to cope with a rich posterior landscape can uncover the structure. If the group is X-ray bright enough, X-ray data may also help to resolve the degeneracy, by pinpointing the group's center of mass, see Fig. 4.6 for X-ray data for PG1115+080. Finally, we show in Sec. 4.2.5 that some implementations in the literature used an inaccurate kinematical prior, mis-modelling the group velocity dispersion by as much as 20%.
- Ref. [BT25]. The estimates of Ref. [BT21b] for the expected mass of the soliton in a large elliptical lens galaxy relied on relaxation time estimates, which may be inaccurate for realistic halos. In Sec. 5.4, we use numerical simulations to more reliably probe the scenario of Ref. [BT21b]. Our simulations mimic galactic halos and explore different initial conditions and levels of sophistication in the modelling of the halo potential. For $m \sim 10^{-25}$ eV, ULDM cores act as approximate internal mass sheets in strong gravitational lensing, and could first be detected using time delays in cosmography, when an H_0 prior is assumed: a scenario we dub AxionH0graphy, see Fig. 5.6 right for a proof of concept with a mock system. The MSD is broken by finite core radius and by the dynamical displacement of cores from the halo center of mass, which introduce imaging distortions and restrict the H_0 signal limit of AxionH0graphy to $m \lesssim 5 \times 10^{-25}$ eV, see Tab. 5.2 for a complete summary of the simulations we considered. Cosmological simulations are called for to sharpen the predicted connection between the amplitude of ULDM galactic cores and the ULDM cosmological fraction.

- Ref. [TCB25]. We present in Sec. 6.1 numerical simulations of dark matter and stellar dynamics in ULDM halos tailored to mimic dwarf galaxies. For dark matter particle mass $m \simeq 10^{-22}$ eV, dynamical heating causes the half-light radius to over-shoot surface brightness data of the Fornax galaxy, see Fig. 6.6. For $m \simeq 10^{-21}$ eV, soliton core formation leads to a velocity dispersion peak incompatible with kinematics data, see Fig. 6.1. Extending the analysis to the Carina and Leo II galaxies, the tension persists up to $m \simeq 5 \times 10^{-21}$ eV, see Fig. 6.2. A caveat in our analysis is the omission of stellar self-gravity. This would not change dynamics today, but could affect extrapolation back in time if the stellar body was more compact in the past.
- Ref. [Blu+25]. The rotation curve observational implications of solitons depend on their mass. Various relations for the ratio of the masses of a soliton and the halo have been proposed. We analyse the implications of these relations in Sec. 6.2, and test them with a suite of numerical simulations. The relation of Ref. [Sch+14] is equivalent to $(E/M)_{\text{sol}} = (E/M)_{\text{halo}}$ where $E_{\text{sol(halo)}}$ and $M_{\text{sol(halo)}}$ are the energy and mass of the soliton (halo). If the halo is approximately virialized, this relation is parametrically similar to the evaporation/growth threshold of Ref. [CSX24], and it thus gives a rough lower bound on the soliton mass. A different relation has been proposed by Ref. [Moc+17], which is equivalent to $E_{\text{sol}} = E_{\text{halo}}$, so is an upper bound on the soliton mass provided the halo energy can be estimated reliably. Our simulations provide evidence for this picture, see Figs. 6.9, 6.11 and are in broad consistency with the literature, in particular after accounting for ambiguities in the definition of E_{halo} at finite volume, see discussion in Sec. 6.3, in particular Fig. 6.17. We also note that the quantity Ξ , often used in the literature to characterize the soliton-halo relation, is badly sensitive to boundary effects, see Figs. 6.13, 6.14.

In the near future, it will be important to refine the theoretical predictions for ULDM in order to better understand how we can constrain (or, if lucky, provide evidence for) the ULDM paradigm. Additionally, forecasts suggest that H_0 measurements from gravitational lensing will achieve the precision and accuracy needed to determine whether the result aligns with the distance ladder measurement or with the early universe/LSS probes [BT21a], offering valuable insights into the resolution of the highly debated H_0 tension. Assessing the accuracy and reliability of such measurements is therefore a timely endeavour.

Appendix A

Solving numerically the Schrödinger–Poisson Equations

A.1 Solving the Schrödinger–Poisson equations

The ULDM for multiple fields follows the SPE for non-relativistic fields ψ_i each with particle mass m_i ; in the possible presence of an external potential Φ_{ext} , those read (Eqs. (3.14), (3.15), which we rewrite here)

$$i\frac{\partial\psi_i}{\partial t} = -\frac{1}{2m_i}\nabla^2\psi_i + m_i(\Phi + \Phi_{\text{ext}})\psi_i, \quad (\text{A.1})$$

$$\nabla^2\Phi = 4\pi G(\sum_i |\psi_i|^2 - \langle \sum_i |\psi_i|^2 \rangle), \quad (\text{A.2})$$

where $\langle \cdot \rangle$ indicates the spatial average.

It is convenient to express the fields and coordinates in a dimensionless form. To this end, we define

$$\tilde{\psi}_i = \frac{1}{\lambda_s^2} \frac{\sqrt{4\pi G}}{m_1} \psi, \quad \tilde{x} = \lambda_s m_1 x, \quad \tilde{t} = \lambda_s^2 m_1 t, \quad \tilde{\Phi} = \Phi / \lambda_s^2, \quad \tilde{\Phi}_{\text{ext}} = \Phi_{\text{ext}} / \lambda_s^2, \quad (\text{A.3})$$

with λ_s a dimensionless real parameter. This gives

$$i\frac{\partial\tilde{\psi}_i}{\partial\tilde{t}} = -\frac{1}{2r_i}\nabla_{\tilde{x}}^2\tilde{\psi}_i + r_i(\tilde{\Phi} + \tilde{\Phi}_{\text{ext}})\tilde{\psi}_i, \quad (\text{A.4})$$

$$\nabla_{\tilde{x}}^2\tilde{\Phi} = \sum_i |\tilde{\psi}_i|^2 - \langle \sum_i |\tilde{\psi}_i|^2 \rangle, \quad (\text{A.5})$$

where we defined

$$r_i := \frac{m_i}{m_1}. \quad (\text{A.6})$$

The density of the i -th field is $\tilde{\rho}_i = |\tilde{\psi}_i|^2$, its total mass is

$$\tilde{M}_i = \int d^3\tilde{x} |\tilde{\psi}_i|^2, \quad (\text{A.7})$$

and the energy associated to each field component reads

$$\tilde{E}_{\text{tot},i} = \int d^3\tilde{x} \left(\frac{|\nabla\tilde{\psi}_i|^2}{2r_i^2} + \frac{1}{2}|\tilde{\psi}_i|^2\tilde{\Phi} + |\tilde{\psi}_i|^2\tilde{\Phi}_{\text{ext}} \right). \quad (\text{A.8})$$

Both $M = \sum_i M_i$ and $E_{\text{tot}} = \sum_i E_{\text{tot},i}$ are conserved under the Schrödinger–Poisson evolution. Under the rescaling in Eq. (A.3),

$$\tilde{M} = \frac{4\pi Gm}{\lambda} M, \quad \tilde{E}_{\text{tot}} = \frac{4\pi Gm}{\lambda^3} E_{\text{tot}} \implies \Xi \equiv \frac{|E_{\text{tot}}|}{M^3} \frac{1}{G^2 m^2} = \tilde{\Xi}. \quad (\text{A.9})$$

Thus, Ξ is invariant under the λ_s rescaling.

In simulations, we solve the rescaled Schrödinger–Poisson equations, in Eqs. (A.4) and (A.5), using a standard pseudo-spectral 3D solver, similar to that described in Ref. [LPT18], in a periodic box with length $\tilde{L}$. In the following, we review the basics of its implementations. The field is evolved via the unitary operator

$$\tilde{\psi}_i(\tilde{t} + d\tilde{t}) = \prod_{\alpha} e^{-ir_i d_{\alpha} d\tilde{t}(\tilde{\Phi}_{\alpha} + \tilde{\Phi}_{\text{ext}})} e^{-ic_{\alpha} d\tilde{t} \frac{(-i\nabla)^2}{2r_i}} \tilde{\psi}_i(\tilde{t}), \quad (\text{A.10})$$

where $d\tilde{t}$ is the time-step. In this equation, first the kinetic operator acts on $\tilde{\psi}(\tilde{t})$ as

$$e^{-ic_{\alpha} d\tilde{t} \frac{(-i\nabla)^2}{2r_i}} \tilde{\psi}_i(\tilde{t}) \equiv \tilde{\psi}_i^{(\alpha)}(\tilde{t} + d\tilde{t}), \quad (\text{A.11})$$

followed by the potential operator, which is obtained by solving:

$$\nabla^2 \tilde{\Phi}_{\alpha} = \sum_i |\tilde{\psi}_i^{(\alpha)}|^2 - \langle |\sum_i \tilde{\psi}_i^{(\alpha)}|^2 \rangle. \quad (\text{A.12})$$

The range that α runs over, and the values of the constants c_{α} , d_{α} , depend on the order of the numerical integrator that is used. We make use of the 6th order version, for which the numerical values of c_{α} , d_{α} can be found in [LPT18]. We evaluate Eqs. (A.11) and (A.12) in Fourier space, via fast Fourier transforms using the FFTW library [Fri99]. We use adaptive time-steps, to ensure a conservation of energy $\Delta E/E \lesssim 10^{-5}$ between time-steps, and an overall conservation of energy $|E_{\text{final}}^{\text{tot}} - E_{\text{initial}}^{\text{tot}}|/|E_{\text{final}}^{\text{tot}} + E_{\text{initial}}^{\text{tot}}| \lesssim 10^{-3}$.

Eq. (A.12) is solved by setting the zero-momentum mode ($k = 0$) of $\tilde{\Phi}$ to zero, implying that the average of $\tilde{\Phi}$ over the box is zero, $\langle \tilde{\Phi} \rangle = 0$. To fix the arbitrary constant in the potential energy E_{pot} , we define

$$\tilde{\Phi}(\tilde{x}) \equiv \tilde{\Phi}_{\text{code}}(\tilde{x}) + c, \quad (\text{A.13})$$

where $\tilde{\Phi}_{\text{code}}$ is the gravitational potential computed as described above and satisfying $\langle \tilde{\Phi}_{\text{code}} \rangle = 0$. Via an appropriate determination of the constant c , we then require $\lim_{\tilde{r} \rightarrow \infty} \tilde{\Phi}(\tilde{r}) = 0$, where $\tilde{r}$ is the distance to the point of maximum density. Specifically, from the Poisson equation, the asymptotic behavior of $\tilde{\Phi}$ in the infinite-volume limit is

$$\tilde{\Phi} \sim -\frac{\tilde{M}}{4\pi\tilde{r}} \text{ for large } \tilde{r}. \quad (\text{A.14})$$

Hence, if $\tilde{R}$ is the furthest point in the grid from the point of maximum density (within which we interpret all the mass $\tilde{M}$ of the box to lie), we can approximately recover the constant c as

$$\tilde{\Phi}(\tilde{R}) = \tilde{\Phi}_{\text{code}}(\tilde{R}) + c = -\frac{\tilde{M}}{4\pi\tilde{R}} \implies c = -\frac{\tilde{M}}{4\pi\tilde{R}} - \tilde{\Phi}_{\text{code}}(\tilde{R}). \quad (\text{A.15})$$

We stress that this method is just a rough estimate, since there is no guarantee that the potential already goes as Eq. (A.14) within the box.

As a check, in Fig. A.1, we plot our soliton-halo data points, from Fig. 6.11 as a function of Ξ , calculated from the total energy E_{tot} including the constant Eq. (A.15) in E_{pot} (with $\tilde{R} = \tilde{L}/2$). Relative to Fig. 6.11 (which is the same plot but with Ξ calculated from the kinetic energy alone), many of the data points are shifted to the left. We believe this discrepancy is likely due to the calculation of the constant in the potential energy via Eq. (A.15) not being accurate enough. On the other hand, with the corrected E_{tot} the

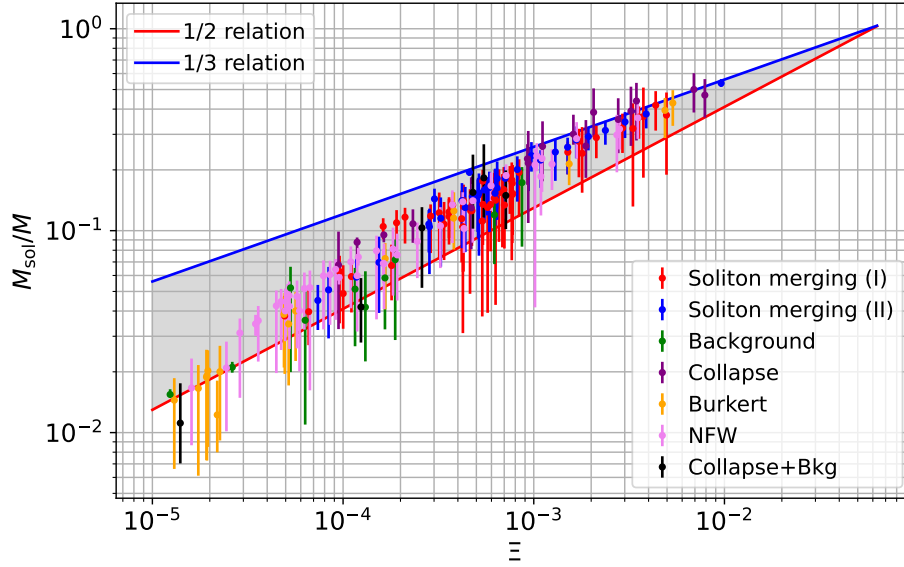


FIGURE A.1: Same as in Fig. 6.11, but with Ξ calculated using E_{tot} instead of E_{kin} . Unlike Fig. 6.17 left, the constant in the potential energy (which enters Ξ) is chosen as in Eq. (A.15) such this matches its infinite-volume value.

points are substantially closer to the results using E_{kin} than when using the uncorrected E_{tot} , as plotted in Fig. 6.17. This suggests that our calculation of the constant in the potential energy is at least partially successful. Given this situation, and the fact that a mis-determination of this constant can affect the interpretation of results, we decided to use quantities calculated from E_{kin} for our main plots in Ch. 6.

A.1.1 Stars

Star dynamics in Sec. 6.1 is computed from the gravitational potential Φ , which is assumed to be sourced only from the ULDM field ψ , neglecting stellar gravity. Star coordinates are evolved via the leapfrog integrator [BT08]

$$\begin{aligned}\vec{r}_i(t_{j+1/2}) &= \vec{r}_i(t_j) + \frac{1}{2} \text{d}t \vec{v}_i(t_j), \\ \vec{v}_i(t_{j+1}) &= \vec{v}_i(t_j) - \text{d}t \nabla \Phi(\vec{r}_i(t_{j+1/2})); \\ \vec{r}_i(t_{j+1}) &= \vec{r}_i(t_{j+1/2}) + \frac{1}{2} \text{d}t \vec{v}_i(t_{j+1/2}).\end{aligned}\tag{A.16}$$

$\nabla \Phi(\vec{r}_i(t_{j+1/2}))$ is computed from the ULDM-sourced Φ in the grid, using multi-linear interpolation.

Notice that one can write multilinear interpolation as

$$f(x, y, z) = \sum_{i,j,k=1,2} f(x_i, y_j, z_k) \left(1 - \frac{|x - x_i|}{x_2 - x_1}\right) \left(1 - \frac{|y - y_j|}{y_2 - y_1}\right) \left(1 - \frac{|z - z_k|}{z_2 - z_1}\right); \tag{A.17}$$

this is the same way the Cloud In Cell (CIC) assigns mass in neighbouring points.

A.2 Soliton solution

The soliton solution can be obtained from Eqs. (A.4) and (A.5) (for a single field, with $\Phi_{\text{ext}} = 0$) with the spherically-symmetric ansatz

$$\tilde{\psi}(\tilde{x}, \tilde{t}) \equiv \chi(\tilde{r}) e^{-i\tilde{\omega}_{\text{sol}} \tilde{t}}, \quad (\text{A.18})$$

which leads to

$$\partial_{\tilde{r}}^2(\tilde{r}\chi) = 2\tilde{r}(\tilde{\Phi} - \tilde{\omega}_{\text{sol}})\chi, \quad (\text{A.19})$$

$$\partial_{\tilde{r}}^2(\tilde{r}\tilde{\Phi}) = \tilde{r}\chi^2. \quad (\text{A.20})$$

These equations are invariant under the λ_s rescaling of Eq. (A.3) and thus admit an infinite class of solutions for χ , labeled by χ_λ . We define them such that $\chi_\lambda(0) = \lambda_s^2$. We can solve the equations above for $\chi_1(0)$ with boundary conditions $\partial_{\tilde{r}}\chi(0) = \partial_{\tilde{r}}\tilde{\Phi}(0) = 0$, $\lim_{\tilde{r} \rightarrow \infty} \chi(\tilde{r}) = 0$ to obtain (see Sec. A.2.1 for further details on the procedure)

$$\tilde{M}_{\text{sol},1} \approx 26, \tilde{r}_{c,1} \approx 1.31, \tilde{\omega}_{\text{sol},1} \approx -0.69 + \tilde{\Phi}(\tilde{r} \rightarrow \infty). \quad (\text{A.21})$$

Here, $\tilde{M}_{\text{sol}} = \int d^3\tilde{x} |\tilde{\psi}|^2 = \int d^3\tilde{x} \chi^2$ is the total soliton mass and $\tilde{r}_c$ is the soliton half-density radius, i.e. $|\tilde{\psi}(\tilde{r}_c)|^2 = |\tilde{\psi}(0)|^2/2$. For an isolated soliton, $\tilde{\Phi}(\tilde{r} \rightarrow \infty) = 0$.¹ A generic soliton will then have

$$\tilde{M}_{\text{sol}} = \lambda \tilde{M}_{\text{sol},1}, \tilde{E}_{\text{sol}} = \lambda^3 \tilde{E}_{\text{sol},1}, \tilde{r}_c = \tilde{r}_{c,1}/\lambda, \tilde{\omega}_{\text{sol}} \approx -0.69\lambda^2 + \tilde{\Phi}(\tilde{r} \rightarrow \infty). \quad (\text{A.22})$$

From these equations it follows $r_c = 1.31/(4\pi G m^2 \rho(0))^{1/4}$ and $M_{\text{sol}}^4 = 64\pi\rho(0)/(G^3 m^6)$, used in Sec 6.2.1.

The soliton is a virialized system, i.e. $2\tilde{E}_{\text{kin,sol}} = -\tilde{E}_{\text{pot,sol}}$,² and one can show that

$$\tilde{E}_{\text{sol}} = \frac{1}{3} \tilde{M}_{\text{sol}} \tilde{\omega}_{\text{sol}} = -\tilde{E}_{\text{kin,sol}}, \quad (\text{A.25})$$

where in the first equality we integrated by parts the equations of motions and used the virial theorem. Hence, since $\tilde{E}_{\text{sol}}/\tilde{M}_{\text{sol}}^3 = \tilde{E}_{\text{sol},1}/\tilde{M}_{\text{sol},1}^3$ given that Ξ is independent of λ , a soliton-halo relation of the form in Eq. (6.22) whose validity is retained in the limit $M_{\text{sol}}/M \rightarrow 1$ should always obey

$$1 = \alpha \left(\frac{|E_{\text{sol},1}|}{G^2 m^2 \tilde{M}_{\text{sol},1}^3} \right)^\beta = \alpha \left((4\pi)^2 \frac{\tilde{\omega}_{\text{sol},1}}{3\tilde{M}_{\text{sol},1}^2} \right)^\beta \approx \alpha (0.054)^\beta. \quad (\text{A.26})$$

This implies that for a soliton one has $\Xi \approx 0.054$. Both $\alpha = 4.2$, $\beta = 1/2$ (from [Sch+14]) and $\alpha = 2.6$, $\beta = 1/3$ (from [Moc+17]) approximately satisfy Eq. (A.26). From this result, one can infer that the 1/2 relation of Ch. 6 is simply rewritten as in Eq. (6.16), whereas the 1/3 relation as in Eq. (6.18).

In the case of a soliton within an external potential Φ_{ext} , the scaling relation becomes

$$\chi_\lambda(\tilde{x}, \tilde{\Phi}_{\text{ext}}) = \lambda^2 \chi_1(\lambda \tilde{x}, \lambda^{-2} \tilde{\Phi}_{\text{ext}}(\lambda^{-1} \cdot)), \gamma_\lambda = \lambda^2 \gamma_1, \tilde{\Phi}_\lambda(\tilde{x}, \tilde{\Phi}_{\text{ext}}) = \lambda^2 \tilde{\Phi}_1(\lambda \tilde{x}, \lambda^{-2} \tilde{\Phi}_{\text{ext}}(\lambda^{-1} \cdot)). \quad (\text{A.27})$$

¹For a soliton surrounded by a halo, one can identify $\tilde{\Phi}(\tilde{r} \rightarrow \infty)$ with $\tilde{\Phi}_{\text{offset}}$ in Eq. (A.144).

²Defining the moment of inertia as

$$I := \frac{1}{2} \int d^3r r^2 |\psi|^2, \quad (\text{A.23})$$

one can show (see App. A of Ref. [Hui+17])

$$\dot{I} = 2E_k + E_{\text{pot}}; \quad (\text{A.24})$$

then, for a soliton, $\dot{I} = 0$, implying $\ddot{I} = 0$, which implies virial theorem.

These cannot be analytically solved, but a good fitting function for the numerical solution is

$$\chi_1(\tilde{x}) = (1 + a^2 \tilde{x}^2)^{-b}. \quad (\text{A.28})$$

For $\Phi_{\text{ext}} = 0$ (i.e. no external potential), one obtains $a \simeq 0.23$, $b \simeq 4.07$.

A.2.1 Further details on solving for the soliton solution

In solving numerically Eqs. (A.19), (A.20), we should avoid the (numerical) singularity at $x = 0$; to do that, we must make the integration start at a $\delta \ll 1$. Since initial conditions are given at $x = 0$, we should consider

$$\chi(\delta) \simeq \chi(0) + \chi'(0)\delta + \frac{1}{2}\chi''(0)\delta^2 + \frac{1}{6}\chi'''(0)\delta^3 + \frac{1}{24}\chi''''(0)\delta^4, \quad (\text{A.29})$$

and similarly for $U(\delta) := \tilde{\Phi}(\delta) - \tilde{\omega}_{\text{sol}}$; the value of $\chi(\delta)$, $U(\delta)$ and their first derivatives $\chi'(\delta)$, $U'(\delta)$ are simply found by plugging the Taylor expansion (A.29) for both functions in Eqs. (A.19), (A.20) and equating coefficients with same powers of δ . As an example, take the Poisson equation, which becomes

$$\begin{aligned} \frac{2}{\delta}U'(0) + 3U''(0) + 2U'''(0)\delta + \frac{5}{6}U''''(0)\delta^2 &= 1 + \chi''(0)\delta^2 \\ \implies U'(0) = U'''(0) = 0, \quad U''(0) = \frac{1}{3}, \quad \chi''(0) &= \frac{5}{6}U''''(0). \end{aligned} \quad (\text{A.30})$$

Using also the other equation in a similar way, one finds

$$U(\delta) = U(0) + \frac{1}{6}\delta^2 + \frac{1}{30}U(0)\delta^4, \quad \chi(\delta) = 1 + \frac{1}{3}U(0)\delta^2 + \frac{1}{60}(1 + 2U^2(0))\delta^4. \quad (\text{A.31})$$

The value for $U(0)$ is found by imposing the ground state no-node solution; to determine the value of $\tilde{\omega}_{\text{sol}}$, we know that, for an sufficiently high R , $\Phi \simeq -MG/R$, and M can be recovered from (3.21), thus yielding

$$U(R) = \tilde{\Phi}(\tilde{R}) - \tilde{\omega}_{\text{sol}} \implies \tilde{\omega}_{\text{sol}} = -U(R) - \frac{1}{\tilde{R}} \int_0^{\tilde{R}} dx x^2 \chi^2(x). \quad (\text{A.32})$$

The above discussion has little changes when we add a Φ_{ext} contribution; the above relations change as (here we ignore higher order than δ^2 terms)

$$\chi''(0) = \frac{2}{3}(U(0) + \tilde{\Phi}_{\text{ext}}(0)), \quad \chi'''(0) = \tilde{\Phi}'_{\text{ext}}(0), \quad \chi''''(0) = \frac{2}{5}(1 + 3\tilde{\Phi}''_{\text{ext}}(0) + 2(U(0) + \tilde{\Phi}_{\text{ext}}(0))^2). \quad (\text{A.33})$$

The (possibly divergent) term $\tilde{\Phi}_{\text{om}}(0)$ is a constant term which can be reabsorbed in U .

A.3 Discrete Fourier transform in the SPE solver

The SPE solver we described previously uses Discrete Fourier Transform (DFT), performed using `fttw` [Fri99]. The DFT of a sequence $x_j, j = 0, \dots, P-1$, is an array $y_{k_f}, k_f = 0, \dots, P-1$

$$y_{k_f} = \sum_{j=0}^{P-1} x_j e^{-2\pi i j k_f / P} =: F_P(x)_{k_f}; \quad (\text{A.34})$$

we here denote with k_f the discrete Fourier mode, to be distinguished with respect to the continuous Fourier mode k . The normalized inverse DFT is then

$$x_j = \frac{1}{P} \sum_{k_f=0}^{P-1} y_{k_f} e^{2\pi i j k_f / P} =: F_P^{-1}(y)_j ; \quad (\text{A.35})$$

notice that

$$\sum_{j=0}^{P-1} e^{2\pi i j (k_f - k'_f) / P} = P \delta_{k_f k'_f} . \quad (\text{A.36})$$

To correctly implement fields in Fourier space in the simulation grid, one should be careful on all the various DFT normalization and conventions. As an example, the 1D inverse Fourier transform of ψ in grid space would be³

$$\psi(x) = \int \frac{dk}{2\pi} e^{ikx} \psi_k = \frac{1}{L} \int dk_f \psi_{k_f} e^{2\pi i j k_f / P} \simeq \frac{1}{L} \sum_{k_f=-P/2}^{P/2-1} \psi_{k_f} e^{2\pi i j k_f / P} = \psi_j = \frac{1}{P} \sum_{k_f=-P/2}^{P/2-1} \psi_{k_f} e^{2\pi i j k_f / P} , \quad (\text{A.37})$$

where we used the relation between continuous and discrete Fourier mode $k = 2\pi k_f / L$, together with $x = Lj / N$. Notice that the chain of equalities in Eq. (A.37) holds only if

$$\psi_k = \frac{L}{P} \psi_{k_f} . \quad (\text{A.38})$$

In n dimensions, it would be

$$\psi_{\vec{k}} = \left(\frac{L}{P} \right)^n \psi_{\vec{k}_f} . \quad (\text{A.39})$$

We remark that $\psi_{\vec{k}}$ is the expression of the continuous version of the Fourier transform of ψ (which can be expressed in grid units), $\psi_{\vec{k}_f}$ is the expression one obtains if one applies the DFT to ψ . In particular, notice that ψ and $\psi_{\vec{k}_f}$ have the same dimensions, whereas $\psi_{\vec{k}}$ does not. If one knows ψ_k , then one can simply write

$$\psi(j) = \frac{1}{L} \sum_{k_f=-P/2}^{P/2-1} \psi_{k_f} e^{2\pi i j k_f / P} . \quad (\text{A.40})$$

Notice that here, both k_f and j can be negative, differently from the standard DFT in Eqs. (A.34),(A.35) (there, the indices k_f, j would correspond to the indices of a P -dimensional array, which conventionally goes from 0 to $P - 1$ in most programming languages). Notice that we can write

$$\psi_{k_f} = \sum_{j=-P/2}^{P/2-1} e^{-2\pi i j k_f / P} \psi_j = e^{i\pi k_f} \sum_{j'=0}^{P-1} e^{-2\pi i j' k_f / P} \psi_{j'-P/2} ; \quad (\text{A.41})$$

equivalently, we can write

$$\psi_{k_f} = \left(\sum_{j=-P/2}^{-1} + \sum_{j=0}^{P/2-1} \right) e^{-2\pi i j k_f / P} \psi_j = \sum_{j=0}^{P-1} e^{-2\pi i j k_f / P} \psi_{\tilde{j}} , \quad \tilde{j} = \begin{cases} j , & j = 0, \dots, P/2 - 1 ; \\ j - N , & j = P/2, \dots, P - 1 . \end{cases} \quad (\text{A.42})$$

Notice that, due to the periodicity of the exponential, $\psi_{k_f} = \psi_{k_f+P}$; taking into account that $F_P(\psi)$ would be an array with k varying from 0 to $P - 1$, we can write

$$\psi_{k_f}^o = F_P(\psi^o)_{k_f} , \quad (\text{A.43})$$

³In this section, we assume all the quantities discussed to be in grid space, hence we will omit the tilde.

where in the following, we will indicate with the superscript $^{\circ}$ ordered-arrays, defined via

$$\psi_j^{\circ} = \psi_{\vec{j}}. \quad (\text{A.44})$$

Same reasoning holds for the inverse DFT⁴,

$$\psi_j^{\circ} = F_P^{-1}(\psi^{\text{Fo}})_j. \quad (\text{A.45})$$

Lensing quantities in the grid. In Ch. 5, we use our multiple field SPE solver to compute relevant lensing quantities associated to the ULDM field. In particular, our simulation outputs convergence on an equally-spaced grid; it is thus natural to compute Ψ and its derivatives via FFT. From Eq. (2.27) (hatted quantities are in Fourier space),

$$\hat{\Psi}_{\vec{k}} = -\frac{2}{k^2} \hat{\kappa}_{\vec{k}}. \quad (\text{A.46})$$

In a grid of length L with P points, with $\vec{j} = (j_1, j_2)$, $j_1, j_2 = 0, \dots, P-1$ labelling the grid point, we can write

$$\Psi(\vec{\theta}) = -\frac{2}{(2\pi)^2} \int d^2k \frac{e^{i\vec{k} \cdot \vec{\theta}}}{k^2} \hat{\kappa}_{\vec{k}} \approx -\frac{2}{(2\pi)^2} \sum_{l_1, l_2=-P/2}^{P/2-1} e^{2\pi i \vec{l} \cdot \vec{j}/P} \frac{\hat{\kappa}_{\vec{l}}}{l^2}, \quad (\text{A.47})$$

where we discretized the integral via $\theta_i = j_i L/P$, $k_i = 2\pi l_i/L$, $l_i = 0, \dots, P-1$. Using DFT, we can write

$$\Psi^{\circ} = -2 \left(\frac{L}{2\pi} \right)^2 F_P^{-1} \left(\frac{F_P(\kappa^{\circ})_{\vec{l}}}{l^2} \right). \quad (\text{A.48})$$

Analogously, we can write, for Ψ derivatives,

$$\partial_i \Psi^{\circ} = -i \frac{L}{\pi} F_P^{-1} \left(F_P(\kappa^{\circ})_{\vec{l}} \frac{l_i}{l^2} \right), \quad (\text{A.49})$$

$$\partial_i \partial_j \Psi^{\circ} = 2 F_P^{-1} \left(F_P(\kappa^{\circ})_{\vec{l}} \frac{l_i l_j}{l^2} \right), \quad (\text{A.50})$$

$$\partial_i \partial_j \partial_m \Psi^{\circ} = i \frac{4\pi}{L} F_P^{-1} \left(F_P(\kappa^{\circ})_{\vec{l}} \frac{l_i l_m l_j}{l^2} \right). \quad (\text{A.51})$$

As application of the previous discussion, useful for the results presented in Ch. 5, in Fig A.2 we show γ_1 and F_1 as proxy for shear and flexion magnitude, computed using Eqs. (A.50)-(A.51). These results refer to the system showed in the bottom-left panel of Fig. 5.10, which has an high $\Delta\kappa_c \gtrsim 0.09$. With a large $\Delta\kappa_c$, the system under consideration can yield possible detectable imaging errors. However, the magnitude of the shear near the center of the core is comparable to (and degenerate with) typical cosmological external shear, which is poorly constrained theoretically. Similarly, the flexion magnitude is not too far off from the flexion caused by a host group of the lens galaxy [TB23]. This calls for the possibility that MSD-breaking effects from cores with $\Delta\kappa_c \gtrsim 0.03$ could still be reabsorbed by other modeling parameters.

⁴Usually, we indicate arrays in Fourier space just with the index k ; when confusion might arise, we will specify it with a superscript $^{\text{F}}$.

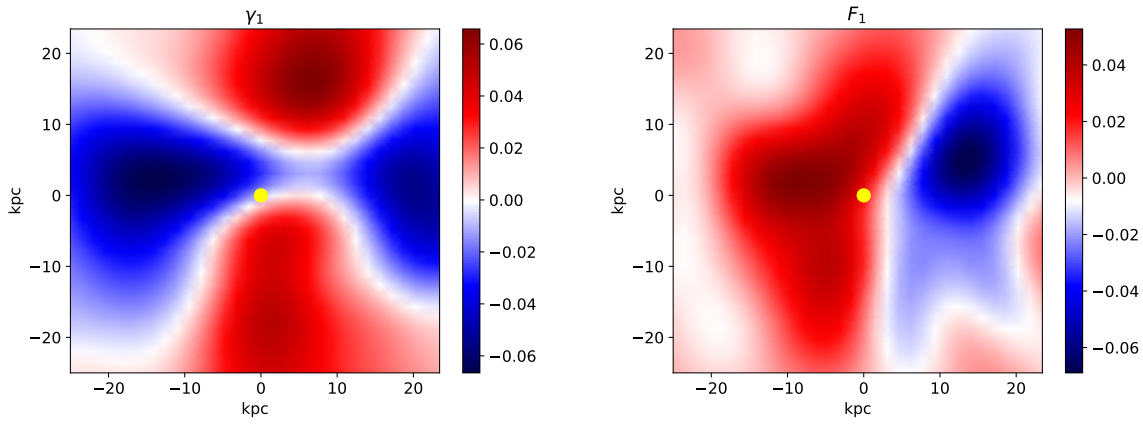


FIGURE A.2: Plots of γ_1 and F_1 on the left and right respectively, referring to the convergence field in the bottom-left panel of Fig. 5.10.

A.4 Some considerations about SPE properties

A.4.1 Center of mass velocity

The velocity vector for a field ψ_i obeying SPE reads

$$\vec{v}_i = \frac{1}{2r_i\rho_i\mathbf{i}}(\psi_i^*\nabla\psi_i - \psi_i\nabla\psi_i^*) ; \quad (\text{A.52})$$

we can write the center of mass velocity $\vec{v}_{\text{cm}}$ as

$$\vec{v}_{\text{cm}} = \frac{1}{M_{\text{tot}}} \int d^3x \sum_i \rho_i \vec{v}_i . \quad (\text{A.53})$$

One can see that $\vec{v}_{\text{cm}}$ does not change with time for the self-gravitating potential Φ (an external potential would change it). In fact (integrating by parts)

$$\begin{aligned} \frac{d\vec{v}_{\text{cm}}}{dt} &= \sum_i \frac{1}{2r_i\mathbf{i}} \int d^3x \left(2\frac{\partial\psi_i^*}{\partial t} \nabla\psi_i - 2\frac{\partial\psi_i}{\partial t} \nabla\psi_i^* \right) = \sum_i \frac{1}{2r_i^2} \int d^3x \underbrace{(-\nabla^2\psi_i^* \nabla\psi_i - \nabla^2\psi_i \nabla\psi_i^*)}_{=0 \text{ after integration by parts}} \\ &\quad - \sum_i \frac{1}{r_i} \int d^3x r_i (\psi_i^* \nabla(\Phi\psi_i) + \psi_i \nabla(\Phi\psi_i^*)) = - \sum_i \int d^3x (\psi_i^* \nabla(\Phi\psi_i) - (\nabla\psi_i)\Phi\psi_i^*) \\ &= - \sum_i \int d^3x \nabla\phi|\psi_i|^2 . \end{aligned} \quad (\text{A.54})$$

Now, recalling

$$\Phi(\vec{x}) = -G \int d^3y \frac{\sum_j |\psi_j(\vec{y})|^2}{|\vec{x} - \vec{y}|} , \quad (\text{A.55})$$

we can write

$$\begin{aligned} \frac{d\vec{v}_{\text{cm}}}{dt} &= G \sum_{i,j} \int d^3x d^3y |\psi_i(\vec{x})|^2 |\psi_j(\vec{y})|^2 \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \\ &= -G \sum_{i,j} \int d^3x d^3y |\psi_i(\vec{x})|^2 |\psi_j(\vec{y})|^2 \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} = 0 , \end{aligned} \quad (\text{A.56})$$

where we noticed that we can safely interchange $x \leftrightarrow y$ and $i \leftrightarrow j$ on the integral, thus obtaining a quantity which is equal to its opposite, hence forced to be zero. In grid units, one can express (using the DFT notation of App. A.3)

$$\vec{v}_{\text{cm}} = \frac{1}{(2\pi)^3 r_i \tilde{M}} \int d^3 k \vec{k} |\tilde{\psi}_{\vec{k}}|^2 \sim \frac{\tilde{L}^3}{r_i P^6 \tilde{M}} \sum_{\vec{k}_f} \frac{2\pi}{L} \vec{k}_f |\tilde{\psi}_{\vec{k}_f}|^2. \quad (\text{A.57})$$

Notice that if ψ is real, $\psi_{\vec{k}}^* = \psi_{-\vec{k}}$ so $\psi_{\vec{k}}^* \psi_{\vec{k}} = \psi_{-\vec{k}} \psi_{-\vec{k}}^*$ and $\vec{v}_{\text{cm}} = 0$ (antisymmetric integrand). As a sanity check, consider the field configuration

$$\psi = A(\vec{x}) e^{i\vec{v} \cdot \vec{x}}, \quad A \in \mathbb{R}; \quad (\text{A.58})$$

we expect $\vec{v}_{\text{cm}} = \vec{v}$. Then

$$M \vec{v}_{\text{cm}} = \int d^3 x d^3 x' A(\vec{x}) A(\vec{x}') \int d^3 k \frac{\vec{k}}{(2\pi)^3} e^{-i\vec{k} \cdot (\vec{x} - \vec{x}')} e^{i\vec{v} \cdot (\vec{x} - \vec{x}')} ; \quad (\text{A.59})$$

now, setting $\vec{y} := \vec{x} - \vec{x}'$, $\int d^3 x' A(\vec{x}') = \int d^3 y A(\vec{x} - \vec{y})$, we have (integrating by parts)

$$\int d^3 y \int d^3 k \vec{k} e^{-i\vec{k} \cdot \vec{y}} f(\vec{y}) = \int d^3 y \int d^3 k i \frac{\partial e^{-i\vec{k} \cdot \vec{y}}}{\partial \vec{y}} f(\vec{y}) = -i(2\pi)^3 \int d^3 y \delta(\vec{y}) \nabla f(\vec{y}), \quad (\text{A.60})$$

with $f(\vec{y}) = A(\vec{x} - \vec{y}) e^{i\vec{v} \cdot \vec{y}}$. We then conclude as expected

$$M \vec{v}_{\text{cm}} = \vec{v} \int d^3 x A^2(\vec{x}) = M \vec{v}. \quad (\text{A.61})$$

A.4.2 Total energy conservation

We can explicitly show that the total energy is conserved. For the kinetic energy (after integration by parts) we have⁵

$$\frac{dE_{\text{kin},i}}{dt} = \frac{1}{2r_i^2} \int d^3 x \left(-\frac{\partial \psi_i}{\partial t} \nabla^2 \psi_i^* - \frac{\partial \psi_i^*}{\partial t} \nabla^2 \psi_i \right) = \frac{1}{2r_i} \int d^3 x \Phi(\psi_i^* \nabla^2 \psi_i - \psi_i \nabla^2 \psi_i^*), \quad (\text{A.62})$$

whereas for the potential energy we have

$$\frac{dE_{\text{pot},i}}{dt} = \frac{1}{4r_i} \int d^3 x \Phi(-\psi_i^* \nabla^2 \psi_i + \psi_i \nabla^2 \psi_i^*) + \frac{1}{2} \int d^3 x |\psi_i|^2 \frac{\partial \Phi}{\partial t}; \quad (\text{A.63})$$

notice that for a time-independent potential, the first term without the factor 1/2 is enough to give the total derivative of the total energy to be zero. For our case, we need

$$\frac{\partial \Phi(x)}{\partial t} = -\frac{G}{2r_i} \int d^3 y \frac{\sum_j (-\psi_j^*(y) \nabla^2 \psi_j(y) + \psi_j(y) \nabla^2 \psi_j^*(y))}{|\vec{x} - \vec{y}|} \quad (\text{A.64})$$

⁵We suppress the tilde in this section, we are always computing quantities in grid units Eq. (A.3)

which implies

$$\begin{aligned} \sum_i \frac{1}{2} \int d^3x |\psi_i|^2 \frac{\partial \Phi}{\partial t} &= \frac{1}{4\pi i} \int d^3y \sum_j (-\psi_j^*(y) \nabla^2 \psi_j(y) + \psi_j(y) \nabla^2 \psi_j^*(y)) \underbrace{\left(-G \int d^3x \sum_i \frac{|\psi_i(x)|^2}{|\vec{x} - \vec{y}|} \right)}_{=\Phi(y)} \\ &= \sum_i \frac{1}{4\pi i} \int d^3x (-\psi_i^* \nabla^2 \psi_i + \psi_i \nabla^2 \psi_i^*) \Phi, \end{aligned} \quad (\text{A.65})$$

hence proving that the total energy, $\sum_i (E_{\text{kin},i} + E_{\text{pot},i})$ is conserved.

A.5 Energy of stars coupled with ULDM

Potential and acceleration field on N stars system. Consider a system of N stars with density distribution

$$\rho(\vec{x}) = \sum_j \rho_j \Theta(R - |\vec{x} - \vec{x}_j|), \quad (\text{A.66})$$

meaning we are considering a system with N balls with radius R and constant density ρ_j , each at position $\vec{x}_j$. We assume R to be small enough so that there is no superposition between different balls. The potential at a point $\vec{x}$ of such a system reads

$$\Phi(\vec{x}) = -G \int d^3y \frac{\rho(y)}{|\vec{x} - \vec{y}|} = \sum_j \Phi_j(\vec{x}), \quad \Phi_j(\vec{x}) = -2\pi G \rho_j \int_0^R y^2 dy \int_{-1}^1 d\cos\theta \sum_{l=0} \frac{y_{<}^l}{y_{>}^{l+1}} P_l(\cos\theta), \quad (\text{A.67})$$

where we are using the linearity of the integral to split it in the integrals over the whole space of a single ρ_j component, and within each integral we use spherical coordinates centered on $\vec{x}_j$. Moreover, we used the expansion

$$\frac{1}{|\vec{x} - \vec{y}|} = \sum_l \frac{y_{<}^l}{y_{>}^{l+1}} P_l(\cos\theta), \quad (\text{A.68})$$

where $y_{>}$ is x if $x > y$, otherwise y if $y > x$, and $y_{<}$ is x if $x < y$, otherwise y if $y < x$. P_l are the Legendre polynomials, for which we need the properties

$$P_0 = 1, \quad \int_{-1}^1 P_l(x) dx = 0 \text{ for } l \geq 1; \quad (\text{A.69})$$

hence, we end up with

$$\Phi_j(\vec{x}) = -4\pi G \rho_j \int_0^R \frac{y^2 dy}{y_{>}} = -4\pi G \rho_j \times \begin{cases} \frac{R^2}{2} - \frac{x^2}{6}, & x < R \\ \frac{R^3}{3x}, & x \leq R. \end{cases} \quad (\text{A.70})$$

Reinstating the coordinates of the centers of the balls, we end up with the final result

$$\Phi(\vec{x}) = -\frac{4\pi G R^3}{3} \sum_j \frac{\rho_j}{|\vec{x} - \vec{x}_j|} = -G \sum_j \frac{m_j}{|\vec{x} - \vec{x}_j|} \text{ if } \vec{x} \text{ outside all balls}, \quad (\text{A.71})$$

$$\Phi(\vec{x}) = -G \sum_{j \neq k} \frac{m_j}{|\vec{x} - \vec{x}_j|} - 4\pi G \rho_k \left(\frac{R^2}{2} - \frac{|\vec{x} - \vec{x}_k|^2}{6} \right) \text{ if } \vec{x} \text{ inside ball labelled with index } k. \quad (\text{A.72})$$

Notice that for $R \rightarrow 0$, one obtains the usual result, without divergences. The acceleration at point $\vec{x}$ will just be

$$-\nabla\Phi = -G \sum_j m_j \frac{\vec{x} - \vec{x}_j}{|\vec{x} - \vec{x}_j|^3} \text{ if } \vec{x} \text{ outside all balls ,} \quad (\text{A.73})$$

$$-\nabla\Phi = -G \sum_{j \neq k} \frac{\vec{x} - \vec{x}_j}{|\vec{x} - \vec{x}_j|^3} + \frac{4\pi G \rho_k}{3} (\vec{x} - \vec{x}_k) \text{ if } \vec{x} \text{ inside ball labelled with index } k . \quad (\text{A.74})$$

Again, no divergent behaviour for $\vec{x} \rightarrow \vec{x}_k$, and the last term vanishes for $R \rightarrow 0$.

As a final exercise, we compute the total potential energy

$$E_p = \frac{1}{2} \int d^3x \Phi(\vec{x}) \sum_j \rho_j \Theta(R - |\vec{x} - \vec{x}_j|) =: \sum_j E_j ; \quad (\text{A.75})$$

for each E_j , we can again use spherical coordinates centered on $\vec{x}_j$ and have

$$E_j = -\frac{2\pi G \rho_j}{2} \int_0^R dx x^2 d \cos \theta \left(\sum_{l \neq j} \frac{m_l}{|\vec{x} - \vec{x}_l|} + 4\pi \rho_j \left(\frac{R^2}{2} - \frac{x^2}{6} \right) \right) =: \sum_{l \neq j} E_{jl} - E_j^{\text{self}} , \quad (\text{A.76})$$

where we used the Φ expression valid for $\vec{x}$ inside the ball j . Using the same multipole expansion procedure we used for Φ , we arrive at

$$E_p = \sum_{l \neq j} E_{jl} + \sum_l E_j^{\text{self}} = -\frac{G}{2} \sum_{l \neq j} \frac{m_j m_l}{|\vec{x} - \vec{x}_l|} - \frac{3}{5R} \sum_j m_j^2 ; \quad (\text{A.77})$$

we see that the self energy diverges for $R \rightarrow 0$, but it is always just a constant, hence harmless.

Energy coupled ULDM and stars system. We want to find the conserved energy for the coupled system ULDM and stars, which obey the equations (we assume the mass of ULDM particle to be 1 for simplicity)

$$i\partial_t \psi = -\frac{\nabla^2 \psi}{2} + (\Phi + \Phi_s) \psi , \quad (\text{A.78})$$

$$\nabla^2 \Phi = 4\pi G |\psi|^2 \implies \Phi(\vec{x}) = -G \int d^3y \frac{|\psi(y)|^2}{|\vec{x} - \vec{y}|} , \quad (\text{A.79})$$

$$\frac{d\vec{v}_j}{dt} = -\nabla (\Phi_s(\vec{x}_j) + \Phi(\vec{x}_j)) , \quad (\text{A.80})$$

$$\Phi_s(\vec{x}) = -G \sum_j \frac{m_j}{|\vec{x} - \vec{x}_j|} , \quad (\text{A.81})$$

where $\Phi_s(\vec{x})$ is the potential sourced by the stars and Φ the one sourced by ULDM. The tentative energy expression is

$$E = \frac{1}{2} \int d^3x |\nabla \psi|^2 + \frac{1}{2} \int d^3x |\psi|^2 (\Phi + \Phi_s) + \frac{1}{2} \sum_j m_j v_j^2 - \frac{G}{2} \sum_{j \neq k} \frac{m_j m_k}{|\vec{x}_j - \vec{x}_k|} + \frac{1}{2} \sum_j m_j \Phi(\vec{x}_j) ; \quad (\text{A.82})$$

however, notice that

$$\frac{1}{2} \sum_j m_j \Phi(\vec{x}_j) = -\frac{G}{2} \sum_j \int d^3x \frac{m_j |\psi(y)|^2}{|\vec{x} - \vec{x}_j|} = \frac{1}{2} \int d^3x |\psi|^2 \Phi_s , \quad (\text{A.83})$$

meaning we can write

$$E = \frac{1}{2} \int d^3x |\nabla \psi|^2 + \frac{1}{2} \int d^3x |\psi|^2 (\Phi + 2\Phi_s) + \frac{1}{2} \sum_j m_j v_j^2 - G \sum_{j \neq k} \frac{m_j m_k}{|\vec{x}_j - \vec{x}_k|}. \quad (\text{A.84})$$

We now compute the total time derivative of all the pieces

$$\frac{dE_{\text{kin}}^{\text{U}}}{dt} = \frac{1}{2i} \int d^3x (\Phi + \Phi_s) (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*), \quad (\text{A.85})$$

$$\frac{dE_{\text{pot}}^{\text{U}}}{dt} = \frac{1}{4i} \int d^3x (\Phi + 2\Phi_s) (-\psi_i^* \nabla^2 \psi + \psi \nabla^2 \psi^*) + \frac{1}{2} \int d^3x |\psi|^2 \left(\frac{\partial \Phi}{\partial t} + 2 \frac{d\Phi_s}{dt} \right); \quad (\text{A.86})$$

in red the terms for ULDM which cancel, as we showed in the previous section. For the stars,

$$\frac{dE_{\text{kin}}^{\text{s}}}{dt} = \sum_j m_j \vec{v}_j \frac{d\vec{v}_j}{dt} = - \sum_j m_j \vec{v}_j (\nabla \Phi(\vec{x}_j) + \nabla \Phi_s(\vec{x}_j)), \quad (\text{A.87})$$

$$\frac{dE_{\text{pot}}^{\text{s}}}{dt} = \frac{G}{2} \sum_{i \neq j} \frac{m_i m_j}{|\vec{x}_i - \vec{x}_j|^3} (\vec{x}_i - \vec{x}_j) (\vec{v}_i - \vec{v}_j); \quad (\text{A.88})$$

terms in blue vanish because relate to the star total energy conservation when they are self-gravitating, as one can show via

$$\nabla \Phi_s(\vec{x}_j) = - \sum_{i \neq j} \frac{G m_i}{|\vec{x}_j - \vec{x}_i|^3} (\vec{x}_j - \vec{x}_i), \quad (\text{A.89})$$

which is equal to the factor in the potential energy, since the $i \neq j$ sum has two terms which are the same (this can be made explicit once you swap $i \leftrightarrow j$). Then, from⁶

$$\nabla \Phi(\vec{x}_j) = G \int d^3x \frac{|\psi|^2}{|\vec{x}_j - \vec{x}|^3} (\vec{x}_j - \vec{x}), \quad \frac{d\Phi_s}{dt} = - \sum_j \frac{G m_j}{|\vec{x}_j - \vec{x}|^3} (\vec{x} - \vec{x}_j) \vec{v}_j, \quad (\text{A.91})$$

one indeed finds the total time derivative of the energy as written in Eq. (A.84) vanishes.

A.6 Initial conditions detail for the ULDM field

In this appendix, we present further details on the initial conditions of our simulations described in Section 6.2.2.

Halos. We initialize halos via the Eddington procedure (see [WK93; Lan+20])

$$\psi(x) = (\Delta v)^{3/2} \sum_{\vec{v}} \sqrt{f(\mathcal{E}(x, v))} e^{im\vec{x} \cdot \vec{v} + i\varphi_{\vec{v}}}, \quad (\text{A.92})$$

⁶If we would have used the expression $\sum_j m_j \Phi(\vec{x}_j)$ instead, we would have needed the result

$$\frac{d\Phi(\vec{x}_i(t))}{dt} = G \sum_j m_j \vec{v}_j \int d^3x \frac{|\psi|^2}{|\vec{x}_i - \vec{x}|^3} (\vec{x}_j - \vec{x}) + \frac{G}{2i} \int d^3x \sum_j \frac{m_j}{|\vec{x} - \vec{x}_j|} (\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*) \quad (\text{A.90})$$

where $\varphi_{\vec{v}}$ is a random phase dependent on $\vec{v}$, Δv is the velocity spacing allowed by resolution in the simulation, and

$$f(\mathcal{E}) = \frac{2}{\sqrt{8\pi^2}} \int_0^{\sqrt{\mathcal{E}}} dQ \frac{d^2\rho}{d\Psi^2}(Q), \quad (\text{A.93})$$

$$\mathcal{E} = \Psi(r) - \frac{v^2}{2}, \quad \Psi := -\Phi + \Phi(r_{\max}), \quad (\text{A.94})$$

where $Q = \sqrt{\mathcal{E} - \Psi}$ and Φ is the overall gravitational potential. As target density, we use a NFW profile [NFW97] or a Burkert profile [Bur15],

$$\rho_{\text{nfw}} = \frac{\rho_0 R_s^3}{r(r+r_s)^2}, \quad \rho_{\text{burk}} = \frac{\rho_0 R_b^3}{(r+R_b)(r^2+R_b^2)}, \quad (\text{A.95})$$

with varying ρ_0 and R_s, R_b .

Elliptical collapse. From the field configuration in Eq. (6.20), we can compute analytically

$$M = \rho_0 R_1 R_2 R_3 \pi^{3/2}, \quad E_{\text{kin}} = \frac{\pi^{3/2} \rho_0}{4m^2} \left(\frac{R_1 R_2}{R_3} + \frac{R_1 R_3}{R_2} + \frac{R_2 R_3}{R_1} \right), \quad (\text{A.96})$$

and

$$\Phi = -(4\pi G) \frac{\pi^{3/2}}{(2\pi)^3} \rho_0 R_1 R_2 R_3 \int d^3k \frac{e^{i\vec{k} \cdot \vec{x} - \sum_i k_i^2 R_i^2 / 4}}{k^2}, \quad (\text{A.97})$$

which implies

$$E_{\text{pot}} = -G \frac{\sqrt{2} \pi^{3/2}}{4} \frac{(\rho_0 R_1 R_2 R_3)^2}{R_3} \int_0^{2\pi} d\varphi \frac{\text{arcsinh} \sqrt{\frac{1-A}{A}}}{\sqrt{1-A}}, \quad (\text{A.98})$$

where $A \equiv \frac{1}{R_3^2} (R_1^2 \cos^2 \varphi + R_2^2 \sin^2 \varphi)$. In the spherical case, $A \rightarrow 1$ and the integral yields just a factor of 2π .

Background. We can satisfy Eq. (6.19) via⁸

$$\psi(x) = \frac{(\Delta k)^3}{L^3} \sum_{\vec{k}} \sqrt{\frac{\hat{C}_{\vec{k}} L^3}{2}} (A_{\vec{k}} + iB_{\vec{k}}) e^{i\vec{k} \cdot \vec{x}}, \quad \hat{C}_{\vec{k}} = \frac{8\bar{\rho} \pi^{3/2}}{k_{\text{bkg}}^3} e^{-k^2/k_{\text{bkg}}^2}. \quad (\text{A.99})$$

Here, L^3 is the volume of the integration (i.e. the simulation volume), $\Delta k = 2\pi/L$ is the resolution in Fourier space, and $A_{\vec{k}}$ and $B_{\vec{k}}$ are uncorrelated Gaussianly-distributed variables with unitary variance and zero mean, satisfying

$$\langle A_{\vec{k}} A_{\vec{k}'} \rangle = \langle B_{\vec{k}} B_{\vec{k}'} \rangle = \delta_{\vec{k}\vec{k}'}, \quad \langle A_{\vec{k}} B_{\vec{k}'} \rangle = 0. \quad (\text{A.100})$$

The kinetic energy per unit volume is

$$\frac{E_{\text{kin}}}{L^2} = \frac{1}{2m^2} \langle |\nabla \psi|^2 \rangle = \frac{1}{2m^2} \int \frac{d^3k}{(2\pi)^3} \hat{C}_{\vec{k}} k^2 = \frac{1}{2m^2} \frac{3\bar{\rho} k_{\text{bkg}}^2}{2} = \frac{3k_{\text{bkg}}^2}{4} \frac{M}{L^3}, \quad (\text{A.101})$$

⁷The phase cannot depend on $\vec{x}$ as well, otherwise the continuity of ψ would be spoiled.

⁸The field in Eq. (A.99) is equivalent to the initialization in [LPT18], which reads $\psi_{\vec{k}} = \sqrt{8\bar{N}} \pi^{3/4} e^{-\vec{k}^2/2} e^{i\varphi_{\vec{k}}}$, where $\varphi_{\vec{k}}$ is uniformly distributed between 0 and 2π , with the identifications $\bar{\rho} = \bar{N}/L^3$ and $k_{\text{bkg}} = 1$.

where $M = \bar{\rho}L^3$. From this, it follows that $E_{\text{kin}}/M = 3k_{\text{bkg}}^2/4m^2$.⁹

Velocity dispersion for background initial conditions. We can compute the average velocity of a field initialized as in Eq. (A.99), in the simplified form

$$C_{\vec{k}} = A\pi^{3/2}l^3 e^{-l^2 k^2/4}, \quad l = \frac{2}{k_{\text{bkg}}}. \quad (\text{A.102})$$

This is, as expected, zero, since

$$\langle \rho \vec{v} \rangle = \frac{1}{2mi} \langle \psi^* \nabla \psi - \psi \nabla \psi^* \rangle = \int \frac{d^3k}{(2\pi)^3} C_k \vec{k} = 0, \quad (\text{A.103})$$

due to the antisymmetric integrand. We can look then at the velocity dispersion

$$3 \langle \rho^2 \rangle \sigma^2 := \langle \rho^2 |\vec{v}|^2 \rangle = -\frac{1}{4m^2} \langle \psi^* (\nabla \psi) \psi^* \nabla \psi + \psi (\nabla \psi^*) \psi \nabla \psi^* - 2(\nabla \psi^*) \psi^* (\nabla \psi) \psi \rangle, \quad (\text{A.104})$$

where the factor 3 is to take into account isotropy in 3 dimensions. It will be insightful to compute before $\langle \rho^2 \rangle$,

$$\begin{aligned} \langle \psi^* \psi^* \psi \psi \rangle &= \frac{L^6}{4} \left(\prod_{i=1}^4 \int \frac{d^3k_i}{(2\pi)^3} \sqrt{C_{k_i}} \right) e^{i(k_1+k_2-k_3-k_4)} \\ &\times \underbrace{\langle (A_{k_1} + iB_{k_1})(A_{k_2} + iB_{k_2})(A_{k_3} - iB_{k_3})(A_{k_4} - iB_{k_4}) \rangle}_{=4(\delta(\vec{k}_1-\vec{k}_3)\delta(\vec{k}_2-\vec{k}_4)+\delta(\vec{k}_1-\vec{k}_4)\delta(\vec{k}_2-\vec{k}_3))(2\pi)^6/L^6} = \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} C_{k_1} C_{k_2} = A^2; \end{aligned} \quad (\text{A.105})$$

the difference with Eq. (A.104) is that, in the last line of the previous, one inserts $\pm ik_i$ factors coming from derivatives (the sign depends on whether the term is complex conjugate or not); the only one which does not vanish due to antisymmetric integrand is coming from the last term in Eq. (A.104), which yields

$$\frac{1}{2m^2} \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} C_{k_1} C_{k_2} \vec{k}_1 \cdot \vec{k}_3 (\delta_{k_1k_3} \delta_{k_2k_4} + \delta_{k_1k_4} \delta_{k_2k_3}) = \frac{3A^2}{m^2 l^2} \implies \sigma = \frac{1}{ml}, \quad (\text{A.106})$$

where we used

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}, \quad \int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}. \quad (\text{A.107})$$

A.6.1 Details of the Eddington implementation

We here go more in depth on the halo field initialization, Eq. (A.92).

Notice that, with ψ initialized as in Eq. (A.92), we recover the correct density and velocity dispersion, upon averaging over phases. In fact,

$$\langle |\psi|^2 \rangle = (\Delta v)^3 \sum_{v,v'} \underbrace{\langle e^{i(\varphi_v - \varphi_{v'})} \rangle}_{=\delta_{vv'}} e^{imx(v-v')} \sqrt{f(x,v)f(x,v')} = (\Delta v)^3 \sum_v f(x,v) \sim \int d^3v f(x,v); \quad (\text{A.108})$$

⁹This differs by a factor of 2 from the more common $\frac{E_{\text{kin}}}{M} = \frac{3}{2} \frac{k_{\text{bkg}}^2}{m^2}$ because the distribution has variance $k_{\text{bkg}}^2/2$ rather than k_{bkg}^2 .

analogously, for the average velocity

$$\begin{aligned}\langle v \rangle &= \frac{1}{M} \int d^3x \frac{1}{2mi} \langle (\psi^* \nabla \psi - \psi \nabla \psi^*) \rangle = \frac{(\Delta v)^3}{M} \int d^3x \sum_{\vec{v}} \vec{v} f(x, v) \\ &\sim \frac{1}{M} \int d^3x \int d^3v \vec{v} f(x, v); \end{aligned} \quad (\text{A.109})$$

notice that $\langle v \rangle = 0$ if f only depends on the modulus of the velocity, which is the case for spherical distributions.

When implementing in grid units the gravitational potential for the computation of $f(\mathcal{E})$, remember that G Newton effectively changes to $\tilde{G} = 1/(4\pi)$. The numerical implementation of Eq. (A.92) reads (notation of App. A.3)

$$\tilde{\psi}(\tilde{x}) = \left(\frac{2\pi(\Delta k_f)}{r_i L} \right)^{3/2} \sum_{\vec{k}_f} \sqrt{\tilde{f}(\tilde{\mathcal{E}}(\tilde{x}, k_f))} e^{i2\pi\tilde{x} \cdot \vec{k}_f / L + i\varphi_{\vec{k}_f}}, \quad (\text{A.110})$$

(recall that $\tilde{x} = L\vec{j}/P$, where P is the number of grid points; k_f is chosen such that $(\Delta k_f) = 1$, but there might be cases where you want to speed up calculations by increasing (Δk_f)).

Notice the conversions¹⁰

$$v = \lambda_s \frac{2\pi}{r_i L} k_f, \quad \frac{d^2 \tilde{\rho}(\tilde{\Psi})}{d\tilde{\Psi}^2} = \frac{4\pi G}{m^2} \frac{d^2 \rho(\Psi)}{d\Psi^2}, \quad (\text{A.112})$$

$$\mathcal{E}(x, v) = \Psi(x) - \frac{v^2}{2} = \lambda_s^2 \left(\tilde{\Psi}(\tilde{x}) - \frac{(2\pi k_f)^2}{2r_i^2 L^2} \right) = \lambda_s^2 \tilde{\mathcal{E}}(\tilde{x}, k_f). \quad (\text{A.113})$$

Notice that the relation

$$v = \lambda_s \tilde{v} = \lambda_s \frac{2\pi}{r_i L} k_f, \quad (\text{A.114})$$

comes from enforcing periodic boundary conditions on

$$e^{im\vec{x} \cdot \vec{v}} = e^{i2\pi\tilde{x} \cdot \vec{k}_f / L} = e^{i2\pi\tilde{j} \cdot \vec{k}_f / P}; \quad (\text{A.115})$$

this will limit the amount of velocity points one can sum $\tilde{f}(\tilde{\mathcal{E}})$ over. Another way of explaining Eq. (A.114) is the following. A grid code will care about the Fourier components

$$\psi_{\vec{k}_f} \propto \sum_{\vec{j}} \sum_{\vec{v}} \sqrt{f(\mathcal{E}(\vec{j}, \vec{v}))} e^{i\varphi_{\vec{v}}} \exp\left(\frac{i\vec{j}}{P} (r\vec{v}L - 2\pi\vec{k}_f)\right); \quad (\text{A.116})$$

¹⁰For v , we can also write

$$\tilde{v}' = \frac{2\pi}{L} k_f, \quad (\text{A.111})$$

i.e. without the r term. The r term will then reappear in the exponent $\exp(i2\pi r\tilde{x} \cdot \vec{k}_f / L)$. Considering that we will have to perform the sum over a grid, it is convenient to remove the r term from the exponent, and this can be done with our choice in Eq. (A.114), which would correspond to the change of variable $\tilde{v} = r\tilde{v}'$ in the sum, with Jacobian $r^{-3/2}$.

hence, from the term in the last exponential, the velocities which satisfy Eq. (A.114) for a certain k_f ¹¹ give the largest contribution, which should thus be included in the sum.

When implementing numerically, the maximum k which makes $f(\mathcal{E})$ not vanish is

$$k_{f,\max} = \sqrt{2\tilde{\Psi}(\tilde{r}_{\min})} \frac{r_i L}{2\pi} = \sqrt{2\Psi(r_{\min})} \frac{L_{\text{phys}} m_i}{2\pi m_1}. \quad (\text{A.118})$$

The scaling of $k_{f,\max}$ with L_{phys} and m_2 can be understood as follows. Increasing L_{phys} would increase the wavelength (hence, decrease the momentum) of a specific integer mode k_f . Hence, the k_f associated to the same physical velocity increases. Likewise, increasing m_i would increase the wavelength in grid space associated to the same physical mode, meaning that it would decrease the physical velocity associated to the same k_f .

A.6.2 Other possible initial conditions

In this section, we analyse other possible initial conditions for the ULDM field ψ , implemented in our pseudo-spectral solver but not used in the main body of this thesis.

Delta initial condition. We can think to initialize ψ with a single Fourier mode component mv_0 as

$$|\psi_k|^2 = N \delta_D(mv_0 - |\vec{k}|). \quad (\text{A.119})$$

The constant N can be connected to the mass M by looking at

$$M = \frac{1}{2\pi^2} \int_0^\infty dk |\psi_k|^2 k^2 = \frac{m^2 v_0^2 N}{2\pi^2} \implies N = \frac{2\pi^2}{m v_0^2} M. \quad (\text{A.120})$$

In grid units, reminding (we use $r := m/m_1$, with m_1 as the reference mass in the multiple fields case)

$$\delta_D(k - mv_0) = \frac{\delta_D(\tilde{k} - r)}{mv_0} \simeq \frac{L \delta_{k_f(Lr/2\pi)}}{2\pi m v_0}, \quad (\text{A.121})$$

$$\int d^3 \tilde{k} \delta(\tilde{k} - r) = 4\pi r^2 = \left(\frac{2\pi}{L}\right)^3 \int d^3 k_f \delta(k_f - Lr/2\pi) \frac{L}{2\pi} \simeq \left(\frac{2\pi}{L}\right)^2 \sum_{\vec{k}_f} \delta_{k_f(Lr/2\pi)} \quad (\text{A.122})$$

(the last term is a sum over Kronecker deltas), we can write

$$\tilde{\psi}_{\vec{k}} = \sqrt{\frac{4\pi G N m_1^2}{v_0} \frac{\pi L}{r}} \delta_{k_f(Lr/2\pi)}. \quad (\text{A.123})$$

We remark that, for the approximation

$$\sum_{\vec{k}_f} \delta_{k_f(Lr/2\pi)} \simeq 4\pi \left(\frac{Lr}{2\pi}\right)^2, \quad (\text{A.124})$$

¹¹Actually, all velocities which satisfy $2\pi n k_f / r_i L$ with $n \in \mathbb{N}$ are the most important; this means that also higher velocities (which would be outside the cutoff $k_f = N - 1$) contribute. This gives the prescription

$$\tilde{\psi}(\tilde{x}) = \left(\frac{2\pi(\Delta k_f)}{r_i L}\right)^{3/2} \sum_{\vec{k}_f}^{k_{f,\max}} \sqrt{\tilde{f}(\tilde{\mathcal{E}}(\tilde{x}, k_f))} e^{i2\pi \vec{f} \cdot \vec{k}_f / P + i\phi_{\vec{k}_f}}, \quad (\text{A.117})$$

where $k_{f,\max}$ can be greater than $N - 1$, and the $\vec{k}_f$ terms in the sum are spaced by Δk_f .

to be good (at $\mathcal{O}(0.1)$), one should count all the points lying on the shell with radius $(Lr/2\pi - 0.5, Lr/2\pi + 0.5)$, and impose $\pi P/L > Lr/2\pi$. The last condition is also required from the fact that you impose the momentum to be non-zero on a defined wavenumber k_f , hence the grid has to have access to such wavenumber.

Heaviside initial condition. Another possible initial condition is

$$|\psi_k|^2 = N\Theta(mv_0 - \vec{k}) , \quad (\text{A.125})$$

which populates all the wavenumbers below mv_0 . With same reasoning as before, one finds $N = \frac{6\pi^2 M}{m^2 v_0^3}$. In grid units,

$$\tilde{\psi}_{\vec{k}} = \sqrt{\frac{24\pi GN}{v_0}} \frac{m_1 \pi \Theta(r - \tilde{k})}{r} = \sqrt{\frac{24\pi GN}{v_0}} \frac{m_1 \pi \Theta(rL/2\pi - k_f)}{r} . \quad (\text{A.126})$$

The approximation

$$\sum_{\vec{k}_f} \Theta(rL/2\pi - k_f) \simeq \frac{4}{3} \pi \left(\frac{rL}{2\pi} \right)^3 , \quad (\text{A.127})$$

is good at $\mathcal{O}(0.01)$ for $rL/2\pi \gtrsim 5$.

A.7 The energy spectrum

To gain deeper insight into the energy distribution of the particles in a ULDM field, one useful tool is the energy spectrum, defined as

$$F(t, \omega) = \frac{dN}{d\omega} . \quad (\text{A.128})$$

This is the number N of ULDM particles with a certain (non-relativistic) energy ω . As the soliton forms, we expect $F(t, \omega_{\text{sol}})$ to increase accordingly. We briefly review relevant features of the energy spectrum.

$F(t, \omega)$ was originally defined in [LPT18] as

$$F(t, \omega) = \int_{-\infty}^{\infty} \frac{dt'}{2\pi} \int d^3x \psi^*(\vec{x}, t) \psi(\vec{x}, t') e^{i\omega(t'-t) - (t-t')^2/\tau^2} , \quad (\text{A.129})$$

where $\Delta\omega \tau \gg 1$, with $\Delta\omega$ the energy resolution and τ a parameter of the Gaussian kernel in the integrand. This expression is justified by considering ψ as a superposition of orthonormal energy eigenmodes ψ_n :

$$\psi = \sum_n c_n(t) \psi_n(\vec{x}) e^{-i\omega_n t} . \quad (\text{A.130})$$

Using $\int d^3x \psi_n^*(\vec{x}) \psi_{n'}(\vec{x}) = \delta_{nn'}$, we have, with $t' - t \equiv \Delta t$,

$$F(t, \omega) = \sum_n c_n^*(t) \int \frac{d\Delta t}{2\pi} c_n(t + \Delta t) e^{i\Delta t(\omega_n - \omega) - \Delta t^2/\tau^2} ; \quad (\text{A.131})$$

if we assume that in the integration range $c_n(t + \Delta t) \approx c_n(t)$, we can complete the square inside the integral to recover a Gaussian integral and obtain

$$F(t, \omega) = \sum_n |c_n(t)|^2 \frac{\tau}{2\sqrt{\pi}} e^{-(\omega - \omega_n)^2 \tau^2/4} \approx \sum_n |c_n(t)|^2 \delta(\omega_n - \omega) . \quad (\text{A.132})$$

Interestingly, the same result can be derived using an alternative, manifestly real, formula:

$$F(t, \omega) = 2 \operatorname{Re} \int_0^\infty \frac{dt'}{2\pi} \int d^3x \psi^*(\vec{x}, t) \psi(\vec{x}, t + t') e^{i\omega t' - t'^2/\tau^2}. \quad (\text{A.133})$$

One can again plug the ansatz of Eq. (A.130) and see that Eq. (A.132) holds. We can also write the previous expression as

$$F(t, \omega) = \int_{-\infty}^\infty \frac{dt'}{2\pi} \int d^3x \psi^*(\vec{x}, t_1) \psi(\vec{x}, t_1 + t') e^{i\omega t' - t'^2/\tau^2}, \quad t_1 = \begin{cases} t - t' & \text{for } t' < 0, \\ t & \text{for } t' \geq 0. \end{cases} \quad (\text{A.134})$$

The formulation Eq. (A.133) is more convenient, since in a simulation one can compute Eq. (A.133) by storing only one snapshot at time t and perform the integral until the next snapshot; this avoids using a large amount of memory storing multiple snapshots.

In the grid, assuming equal time-steps with spacing Δt , we have

$$\tilde{F} = \sum_n |c_n(\tilde{t})|^2 \frac{\Delta \tilde{t}}{\pi} \sum_{l=0}^\infty e^{-(l\Delta \tilde{t})^2/\tilde{\tau}^2} \cos(l\Delta \tilde{t}(\tilde{\omega} - \tilde{\omega}_n)), \quad \tilde{\omega} = \frac{\omega}{\lambda_s^2 m}, \quad \tilde{F} = 4\pi G m \lambda_s F. \quad (\text{A.135})$$

In particular, this function is periodic under

$$\tilde{\omega} \rightarrow \tilde{\omega} + \frac{2\pi p}{\Delta \tilde{t}}, \quad p \in \mathbb{Z}. \quad (\text{A.136})$$

The numerically computed $\tilde{F}$ then makes sense only in a range smaller than the period $2\pi/\Delta \tilde{t}$. For Eq. (A.135) to reduce to

$$\tilde{F} \rightarrow \frac{1}{\pi} \int_0^\infty dt' e^{-(t')^2/\tau^2} \cos(t'(\tilde{\omega} - \tilde{\omega}_n)) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{(\tilde{\omega} - \tilde{\omega}_n)^2}{2\sigma^2}}, \quad \sigma \equiv \frac{\sqrt{2}}{\tau}, \quad (\text{A.137})$$

we need to impose

$$\Delta \tilde{\omega} \tau \gg 1, \quad \Delta \tilde{\omega} \Delta \tilde{t} \ll 1, \quad \frac{\tau}{\Delta \tilde{t}} \ll 1, \quad (\text{A.138})$$

where $\Delta \tilde{t}$ is the time between snapshots.

The dimensionless c_n coefficients relate to the number of particles in state n ,

$$\tilde{F} = \frac{d\tilde{N}}{d\tilde{\omega}} \sim |c_n|^2 \delta(\omega - \tilde{\omega}_n). \quad (\text{A.139})$$

One can see that, from Eq. (A.134), we have

$$\int d\tilde{\omega} \tilde{F} = \tilde{M}, \quad \tilde{E}_{\text{part}} \equiv \int d\tilde{\omega} \tilde{\omega} \tilde{F}, \quad (\text{A.140})$$

where E_{part} is the sum of the energy of all the particles in the box. Working out the last equality, one finds out that $\tilde{F}$ encodes the particle energy, where the energy is the sum of the kinetic energy and the contribution of the gravitational potential, the latter viewed as an external potential. In particular, $\tilde{E}_{\text{part}} \neq \tilde{E}_{\text{tot}}$, where $\tilde{E}_{\text{tot}}$ is the total energy of the system, where the fact that the gravitational potential is self-sourced is taken into account. In fact,

$$\int d\omega \omega e^{i\omega t'} = -2\pi i \frac{\partial}{\partial t'} \delta(t') \implies \int d\omega \omega F = \int d^3x \psi^* i \frac{\partial \psi}{\partial t} = \int d^3x \frac{|\nabla \psi|^2}{2} + \int d^3x \Phi |\psi|^2, \quad (\text{A.141})$$

without a factor of 1/2 on the potential energy contribution.

Notice that $\tilde{\omega} \sim \tilde{E}_{\text{kin}} + \tilde{\Phi}$, in particular a constant shift in $\tilde{\Phi}$ would yield a constant shift in $\tilde{\omega}$; from Eq. (A.15), we can write

$$\tilde{F}_{\text{code}}(\tilde{\omega}_{\text{code}}) = \tilde{F}(\tilde{\omega}), \quad \tilde{\omega} = \tilde{\omega}_{\text{code}} + c, \quad (\text{A.142})$$

where $\tilde{F}_{\text{code}}, \tilde{\omega}_{\text{code}}$ are the quantities coming from our Schrödinger–Poisson solver, and c is the constant in Eq. (A.13). When a halo is present, ω_{sol} gets shifted due to the further potential well coming from the halo. To understand the origin of such a shift, one can look at the gravitational potential for a spherical distribution:

$$\Phi(r) = -\frac{G(M_{\text{sol}}(r) + M_{\text{halo}}(r))}{r} - 4\pi G \int_r^\infty dy y (\rho_{\text{sol}}(y) + \rho_{\text{halo}}(y)). \quad (\text{A.143})$$

We further approximate $\rho_{\text{halo}}(r) \approx 0$ for $r < r_t$ and $\rho_{\text{sol}}(r) \approx 0$ for $r > r_t$, where r_t is a transition radius (which can be taken to be $3r_c$). Then, for $r < r_t$,

$$\Phi(r) = \Phi_{\text{sol}}(r) - 4\pi G \int_{r_t}^\infty dy y \rho_{\text{halo}}(y) \equiv \Phi_{\text{sol}}(r) + \Phi_{\text{offset}}, \quad (\text{A.144})$$

and hence

$$\tilde{\omega}_{\text{code}}^{\text{sol}} + c = \tilde{\omega}_{\text{no halo}}^{\text{sol}} + \tilde{\Phi}_{\text{offset}}, \quad (\text{A.145})$$

where again $\tilde{\omega}_{\text{code}}^{\text{sol}}$ is the location of the soliton energy as coming out from our Schrödinger–Poisson solver, while $\tilde{\omega}_{\text{no halo}}^{\text{sol}}$ is the actual energy corresponding to an isolated soliton.

In Fig. A.3 we show an example of $\tilde{F}$ for the snapshots of the simulation plotted in Fig. 6.15. One can see the formation of the soliton via a peak arising at $\tilde{\omega}_{\text{no halo}}^{\text{sol}} + \tilde{\Phi}_{\text{offset}}$. The computation of $F(\omega, t)$ is an interesting cross-check of our main method adopted for defining the moment of soliton formation, Eq. (6.34). However, using $F(\omega, t)$ there remain uncertainties, in particular when to call the forming peak a soliton. These mean that this method of identifying the formation time does not have a clear advantage compared to simply considering the density profile.

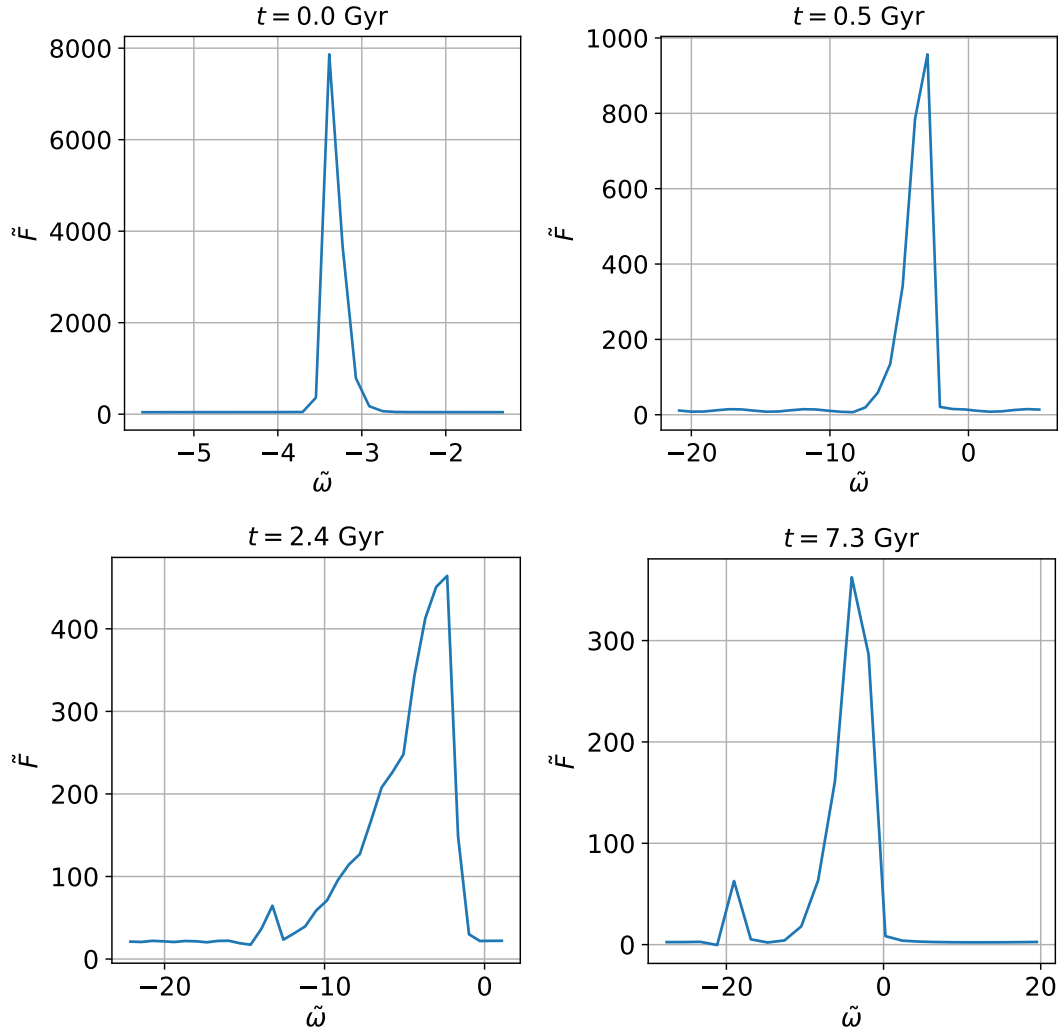


FIGURE A.3: The spectrum $\tilde{F}(\tilde{\omega})$ for the same simulation and times as plotted in Fig. 6.15. One can see a peak forming (corresponding to the soliton) and separate from the main peak corresponding to the halo. In particular, at $t = 7.3$ Gyr the peak lies at around $\tilde{\omega} \approx -19$; integrating the area in the vicinity of that peak, one finds $\tilde{M}_{\text{sol}} \approx 100$, consistent with the soliton mass obtained by fitting of the radially-averaged density profile. Notice that an isolated soliton of that mass would have $\tilde{\omega} \approx -10$. The difference is due to Φ_{offset} , Eq. (A.144).

Appendix B

Gravitational Lensing in the Kottler Metric

At the moment of writing, the following is part of a work in progress, with B. Zhaoyu and K. Blum. The computation of the time delays as performed in Sec. B.4 is original, not seen in the literature anywhere as far as we are aware.

When dealing with typical gravitational lens systems, we have a separation of scales between the cosmological angular diameter distances lens-observer D_l , observer-source D_s and lens-source D_{ls} , order Gpc, and the extension of the lens, order 100 kpc for galaxies. This suggests employing the approximation we outlined in Sec. 2.1, where one splits the cosmological background contribution and the strong perturber one (see discussion around Eq. (2.9)).

For most gravitational lensing applications, e.g. quasar time delay cosmography [Suy+17; Mil+20], this approximation is more than justified. However, when dealing with precision lensing angle measurements, or precise time-delay measurements (see e.g. the proposal in [WSP21]), such an approximation needs more scrutiny [Sas93; FK11; MFM13; Zho+24]. Moreover, from a theoretical point of view, it is interesting to understand from which zeroth-order Taylor expansion the approximation comes from.

The most simple setting to answer these questions comes from the Kottler metric. The Kottler metric [Kot18] (see e.g. [LR77; Per04] for more recent works) is an exact solution of the Einstein equations, with a point mass and a cosmological constant Λ . It is also called Schwarzschild-de Sitter metric for $\Lambda > 0$ and Schwarzschild-anti de Sitter metric for $\Lambda < 0$. Such a metric is without a doubt an approximation of realistic gravitational lens systems, where cosmology is not de-Sitter and the lens is not just a point mass. However, the Kottler metric has the advantage of being simple enough so that it can be tackled analytically and can give insights on possible effects coming from Λ beyond the purely cosmological ones. Due to these features, it does not come as a surprise that such a metric was studied in the gravitational lensing literature, but most of the efforts focused on the light deflection equation. In particular, there has been a debate whether Λ effects are there in the light deflection equation or not [Pia16; RI07; IR10; ABB17; HHB22; Ser09; AK12]. As we will show in Sec. B.2, we argue in favour of no appreciable Λ effects on the lensing equation, in agreement with the findings of [But16].

The effects of Λ in the time delay computations were not studied with the same care as for the deflection angle. Ref. [KKL06] discusses the Kottler metric effects, among which the Shapiro time delay, in the solar system, not generalizing to cosmological settings. Ref. [SZ08] shows a time delay computation applied to a typical galaxy-galaxy lensing system, however the time delay was not computed as seen by a comoving observer, as we should be considered in this setting. To remedy to this caveat, and to be able to relax the de Sitter assumptions to a more realistic FLRW universe, one can employ the McVittie metric [McV33; KKM10], which is an exact solution of the Einstein equations in the presence of a cosmological background density and a point mass (of which the Kottler metric in isotropic coordinates is a special case), to understand the effects for the deflection angle [Par08; Pia16; BP22]. Ref. [Ara11] uses the McVittie metric to compute Shapiro time delays in systems like our solar system, but applications to cosmological systems are not discussed.

To relax the point mass assumption to a more realistic spherically symmetric solution, one can employ the Einstein-Straus metric [ES45; Sch54], also known as Swiss-cheese models. Some works attempted lensing computations in this metric [Sch09], time delays included [BGZ11; CKD10]. Arguably, an Einstein-Straus metric gets closer to physical systems at the expense of much bigger complications on the analytic side.

Here, we stick with the Kottler metric, keeping in mind galaxy-galaxy lens systems. We show that there is no need to use the McVittie metric to compute the time delays as measured by a comoving observer, even *in static coordinates*. We show that, up to order $r_{\text{Sc}}\mathcal{O}(\theta)$, where $\theta \sim 10^{-5}$ rad is the typical observed angle size and r_{Sc} the Schwarzschild radius, cosmology enters into the time delay only via angular diameter distances, and the usual lens equation Eq. (4.13) and time delays Eq. (B.21) are recovered as the leading order expansion in the small quantity θ . Nevertheless, one should be aware of the higher-order corrections we show, which can become relevant for settings such as in Ref. [WSP21].

We show the main, exact equations for the deflection of light and time delays in Sec. B.1; we employ a small-angle approximation and expand over θ , in Sec. B.2 for the light deflection, in Sec. B.3 for the redshift and in Sec. B.4 for the time delay, where we show our novel result.

B.1 Light deflection and time delays in the Kottler metric

The Kottler metric in static coordinates reads

$$ds^2 = -A(r) dt^2 + A^{-1}(r) dr^2 + r^2 d\Omega^2, \quad A(r) = 1 - \frac{r_{\text{Sc}}}{r} - H^2 r^2, \quad (\text{B.1})$$

where r_{Sc} is the Schwarzschild radius and H^2 is the Hubble constant (connected to the cosmological constant Λ with $H^2 = \Lambda/3$). We limit our discussion to the case $\Lambda > 0$. These coordinates explore the causal diamond centered at the point mass in $r = 0$. Horizons arise at the radius r_{hor} which satisfies

$$A(r_{\text{hor}}) = 0 \implies H^2 \frac{r_{\text{hor}}^3}{r_{\text{Sc}}} - \frac{r_{\text{hor}}}{r_{\text{Sc}}} + 1 = 0; \quad (\text{B.2})$$

in the regime $r_{\text{Sc}}H \ll 1$, the previous has two real solutions, $r_{\text{hor}} \simeq r_{\text{Sc}}$ and $r_{\text{hor}} \simeq 1/H$, which are the expected event horizon and cosmological horizon respectively. Static coordinates are well-defined only between these two horizons.

The geodesic equations for the metric Eq. (B.1) can be found by building the Lagrangian density (we denote $\dot{} := d/d\lambda$, where λ is the affine parameter describing the geodesic)

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\frac{1}{2} A(r) \dot{t}^2 + \frac{\dot{r}^2}{2A(r)} + \frac{r^2}{2} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta); \quad (\text{B.3})$$

from the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial x^\mu} = \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}, \quad (\text{B.4})$$

and the normalization condition $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = e$, where $e = 0, e < 0$ for massless and massive geodesics respectively, we obtain

$$\dot{t} = \frac{E}{A}, \quad (\text{B.5})$$

$$\dot{\varphi} = \frac{b}{r^2}, \quad (\text{B.6})$$

$$\dot{r} = \pm \sqrt{Ae + E^2 - \frac{Ab^2}{r^2}}. \quad (\text{B.7})$$

In particular, Eqs. (B.5),(B.6) are obtained by exploiting the fact that metric coefficients do not depend on t and φ , with E and b the respective constants of motion. We exploit the spherical symmetry of the problem to set $\theta = \pi/2$ ¹. The sign in Eq. (B.7) should be chosen according to whether the geodesic has decreasing or increasing r . Notice that it is possible to rescale away E by redefining the affine parameter λ in the massless case. Hence, we will always set $E = 1$ for massless geodesics. This is not possible in the massive case. However, for comoving observers, $E = 1$ is a natural choice, as explained in Ref. [But16]. Deviations from $E \neq 1$ can be interpreted as peculiar velocities of the massive geodesic, as we will show in the following. We will consider near-comoving observers, where E deviations from 1 are small. We normalize the massive geodesic so that $e = -1$. With such a choice, λ will be the massive geodesic proper time.

We now want to describe a strong gravitational lensing setting in this metric, where light rays coming from the same near-comoving source get deflected by the point mass at the origin and reach a near-comoving observer. We denote with ω_i^μ the 4-velocity of the i -th light ray from the source at $x_s = (t_s, r_s, \pi/2, \varphi_s)$ to the observer at $x_{oi} = (t_{oi}, r_{oi}, \pi/2, \varphi_o)$; we assume $\varphi_s \sim \pi > \varphi_o \sim 0$. x_{o1} will be the coordinates of the first observed light-ray event, whereas x_{o2} will pertain the second light-ray event. Notice that r_{oi} can differ for different light rays reaching the observer at different times, even if they come from the same event x_s . u_s will be the 4-velocity of the source and u_o the 4-velocity of the observer. We assume that both source and observer move along lines with constant φ , hence with $\dot{\varphi} = 0$. We have

$$\omega_i^\mu = \left(\frac{1}{A(r)}, \pm \sqrt{1 - \frac{b_i^2 A(r)}{r^2}}, 0, \frac{b_i}{r^2} \right); \quad (\text{B.8})$$

$$u_s^\mu = \left(\frac{E_s}{A(r)}, \sqrt{E_s^2 - A(r)}, 0, 0 \right); \quad (\text{B.9})$$

$$u_o^\mu = \left(\frac{E_o}{A(r)}, \sqrt{E_o^2 - A(r)}, 0, 0 \right). \quad (\text{B.10})$$

The sign on $\dot{r}$ is always positive for massive observers since we assume them to move away from the $r = 0$ singularity, whereas the sign for the massless geodesics will have to be chosen according to whether the light ray is before (minus) or after (plus) the closest approach distance to the point mass. We define $b_i =: s_i |b_i|$, where s_i is the sign of b_i , $s_i = -1$ if the light moves on the orbit plane in a clockwise direction, $s_i = 1$ otherwise. The closest approach distance r_{ci} can be obtained via

$$\dot{r} = 0 \implies (1 + H^2 b_i^2) r^3 - r b_i^2 + b_i^2 r_{Sc} = 0 \implies r^3 - r \tilde{b}_i^2 + \tilde{b}_i^2 r_{Sc} = 0, \quad \tilde{b}_i = \frac{b_i}{\sqrt{1 + H^2 b_i^2}}; \quad (\text{B.11})$$

the smallest positive root of the previous defines r_{ci} .

¹The geodesic motion should be contained in a spatial 2D plane due to the spherical symmetry of the metric; for 3 non-collinear points, as we generically assume here, there pass only one plane, and only two images form. In the limiting case of collinear emitter-black hole observer, then an Einstein ring forms, where two images form in each possible plane passing through the line emitter-lens-observer.

We can now write an expression for the angle deflection (it will be convenient to define the adimensional inverse radius $u = r_{\text{Sc}}/r$, $dr = -r^2 du / r_{\text{Sc}}$)

$$\begin{aligned} \int_{r_s}^{r_{oi}} \left(\frac{d\varphi}{dr} \right)_{\omega_i^\mu} dr &= \varphi_o - \varphi_s + \pi(1 + s_i) = - \int_{r_s}^{r_{ci}} \frac{b_i dr}{r \sqrt{r^2 - b_i^2 A(r)}} + \int_{r_{ci}}^{r_{oi}} \frac{b_i dr}{r \sqrt{r^2 - b_i^2 A(r)}} \\ &= s_i \left(\int_{u_s}^{u_{ci}} + \int_{u_{oi}}^{u_{ci}} \right) \frac{du}{\sqrt{\frac{r_s^2}{b_i^2} - u^2(1-u)}}. \end{aligned} \quad (\text{B.12})$$

Notice that the result we obtain is the same as one would obtain in pure Schwarzschild, the only difference being in the re-definition of the parameter $b_i \rightarrow \tilde{b}_i$. As we will see soon, near-comoving observer effects will not change this conclusion up to the next to leading order, meaning we agree with the findings of [But16].

The deflection angle equation Eq. (B.12) contains the unknown parameters b_i , which are connected to the observable angle in the sky, θ_i . Suppose an observer measures such angles with respect to a light ray, supposedly coming from $\varphi = \varphi_o$ on the lens galaxy. To such light ray we assign the geodesic vector

$$n^\mu = \left(\frac{1}{A(r)}, 1, 0, 0 \right); \quad (\text{B.13})$$

then, an invariant formula for the angle θ_i formed from the light ray n^μ and ω_i^μ with respect to an observer u_o^μ reads

$$1 - \cos \theta_i = - \frac{g_{\mu\nu} \omega_i^\mu n^\nu}{(g_{\mu\nu} u_o^\mu n^\nu)(g_{\mu\nu} u_o^\mu \omega_i^\nu)} \Big|_{x_{oi}} = \frac{(1 - \sqrt{1 - b_i^2 A(r_{oi})/r_{oi}^2})(E_o + \sqrt{E_o^2 - A(r_{oi})})}{E_o - \sqrt{1 - b_i^2 A(r_{oi})/r_{oi}^2} \sqrt{E_o^2 - A(r_{oi})}}. \quad (\text{B.14})$$

As we mentioned earlier, the physical interpretation of E_o is related to the peculiar velocity of the near-comoving observer u^μ with respect to a comoving observer. We can define such peculiar velocity of u^μ with respect to a comoving observer v^μ via

$$\begin{aligned} \frac{1}{\sqrt{1 - v_o^2}} &:= -u_o^\mu v_\mu, \quad v^\mu = \left(\frac{1}{A(r)}, \sqrt{1 - A(r)}, 0, 0 \right) \\ \Rightarrow \frac{1}{\sqrt{1 - v_o^2}} &= \frac{E_o - \sqrt{1 - A(r_{o1})} \sqrt{E_o^2 - A(r_{o1})}}{A(r_{o1})}. \end{aligned} \quad (\text{B.15})$$

E_o seemingly depends on r_{o1} ; notice however that E_o is a constant of motion, meaning that, if Eq. (B.15) is computed along a different r_{o1} , the value of v_o would change accordingly to keep E_o constant. In the following, we assume v_o is the velocity as seen by the observer in r_{o1} . The observer at the slightly different r_{o2} will actually have a different v_o . We will ignore this effect in the following². Notice that when $H = 0$ and $r_{\text{Sc}} = 0$, $-u^\mu v_\mu = E_o$, and we have the simple result

$$\cos \theta_i = \frac{\cos(\theta_i(v_o = 0)) - v_o}{1 - v_o \cos(\theta_i(v_o = 0))}, \quad \cos(\theta_i(v_o = 0)) = \sqrt{1 - \frac{b_i^2}{r_{oi}^2}}, \quad (\text{B.16})$$

where $\theta_i(v_o = 0)$ is the observed angle in absence of peculiar velocities, and the first equation is the aberration of light relation coming from a non-zero v_o .

²The difference $v_{o2}/v_{o1} - 1$ will result to be $\mathcal{O}(x^2)$ in the notation of Sec. B.2, as we show in footnote 7

We are however primarily interested in the time delay between image 1 and 2 of two light rays emitted from the same point x_e ; since we assume image 1 to come before image 2, we can set $t_{o1} = t_0$, i.e. the time corresponding to today³. Denoting the proper time of the observer as T , we can use

$$\dot{r} =: \frac{dr}{dT} \implies \Delta T := T_{o2} - T_{o1} = \int_{r_{o1}}^{r_{o2}} \frac{dr}{\sqrt{E_0^2 - 1 + r_{Sc}/r + H^2 r^2}}. \quad (\text{B.17})$$

Another observable of this setting are the redshifts of the lens and of the emitter. An invariant expression for the observed redshift from a light ray ω_i^μ coming from a source u_s^μ at x_s as seen by an observer u_o^μ at x_o reads

$$1 + z_s := \frac{g_{\mu\nu}(x_s)\omega^\mu(x_s)u_s^\nu(x_s)}{g_{\mu\nu}(x_o)\omega^\mu(x_o)u_o^\nu(x_o)} = \frac{A(r_o)}{A(r_s)} \left(\frac{E_s + \sqrt{E_s^2 - A(r_s)}\sqrt{1 - b^2 A(r_s)/r_s^2}}{E_o - \sqrt{E_o^2 - A(r_o)}\sqrt{1 - b^2 A(r_o)/r_o^2}} \right). \quad (\text{B.18})$$

We can also write an expression for the lens redshift, to be computed using a light ray originating from the lens itself. To simplify our life, we can assume that the light ray used to measure the lens redshift is emitted with $b = 0$, from an emitter sitting at r_{lens} with 4-velocity

$$u_1^\mu = \left(\frac{E_1}{A}, \sqrt{E_1^2 - A - A \frac{b_1^2}{r^2}}, 0, \frac{b_1}{r^2} \right), \quad (\text{B.19})$$

where b_1 relates to u_1^μ and not to the light ray which has $b = 0$. The observer cannot be considered as near-comoving if it sits near the lens. If for example u_1^μ describes a circular orbit at r_{lens} , E_1^2 is determined by the condition $\dot{r} = 0$ at r_{lens} , in particular $E_1^2 = A(r_{\text{lens}})(1 + b_1^2/r_{\text{lens}}^2)$. In any case, the lens redshift would read

$$1 + z_l = \frac{E_o + \sqrt{E_o^2 - A(r_o)}}{A(r_{\text{lens}})} (E_1 - J(r_{\text{lens}})), \quad J(r) := \sqrt{E_1^2 - A(r)} \left(1 + \frac{b_u^2}{r^2} \right). \quad (\text{B.20})$$

Eq. (B.12), Eq. (B.14), Eq. (B.17), Eq. (B.20) and Eq. (B.18) are seven equations⁴ for the observables ΔT , θ_i , the unknowns $\tilde{b}_1$, $\tilde{b}_2$ and the parameters r_s , r_{o1} , $\varphi_o - \varphi_s$, E_o , E_s (connected to the set-up) and r_{Sc} , H (connected to the metric). As we will discuss in more details later, $\tilde{b}_i$ can be directly connected to the observable angles θ_i ; r_e , r_{o1} , E_o and E_e are connected to lens and emitter redshift. The deflection equations Eq. (B.12) can then determine r_{Sc} and $\varphi_o - \varphi_s$, whereas H can be determined via Eq. (B.17).

The quantity r_{o2} does not need to be specified; in fact, notice that we can use the geodesic ω_i^μ to compute the time travel of the light ray as

$$\begin{aligned} \int_{r_s}^{r_{o2}} \left(\frac{dt}{dr} \right)_{\omega_2^\mu} dr - \int_{r_s}^{r_{o1}} \left(\frac{dt}{dr} \right)_{\omega_1^\mu} dr &= t_{o2} - t_{o1} = - \int_{r_s}^{r_{c2}} \frac{dr}{A(r)\sqrt{1 - b_2^2 A(r)/r^2}} \\ &+ \int_{r_{c2}}^{r_{o2}} \frac{dr}{A(r)\sqrt{1 - b_2^2 A(r)/r^2}} - (2 \leftrightarrow 1); \end{aligned} \quad (\text{B.21})$$

this equation introduces another unknown $t_{o2} - t_{o1}$, the static time delay. We can use the geodesic of the observer u_o^μ to compute the static coordinate time elapsed from the time of emission of the light ray to

³When dealing with angular diameter distances, we can normalize comoving distances at t_{o1} ; in the FLRW language, it would correspond in setting $a(t_{o1}) = 1$, where a is the cosmological scale factor, as we will show in Sec. B.3.

⁴These become 9 if we assume we are sensitive to the redshift differences at different times, i.e. if we can distinguish z_{s1} with z_{s2} and same for z_l .

the time the light ray is received. This will give us the final equation we need,

$$t_{o2} - t_{o1} = \int_{r_{o1}}^{r_{o2}} \left(\frac{dt}{dr} \right)_{u_0^\mu} dr = \int_{r_{o1}}^{r_{o2}} \frac{E_0 dr}{(1 - r_{Sc}/r - H^2 r^2) \sqrt{E_0^2 - 1 + r_{Sc}/r + H^2 r^2}}. \quad (B.22)$$

In order to proceed further with the light deflection and time delay computation, we will need to use the small angle approximation, together with expansion on small peculiar velocities.

B.2 Deflection angle in the small angle approximation

In the previous section, we wrote all the exact equations we need for our problem, Eqs. (B.12), (B.14), (B.18), (B.17), (B.21), (B.22). If we want to make some progress, we need to start making some approximations. We can rewrite Eq. (B.11) as (we suppress the i subscript in this section)

$$u^2(1 - u) = \frac{r_s^2}{\tilde{b}^2}. \quad (B.23)$$

We discuss the regime $r_{Sc}/|\tilde{b}| \ll 1$; to connect with physical settings, the Schwarzschild radius associated to a mass $M \simeq 10^{13} M_\odot$ (a massive elliptical) is order parsec, whereas typical angle separation in the same systems is order kpc. In all the expansions in the following, we will denote

$$x := \frac{r_{Sc}}{r_c} \sim \theta \sim \mathcal{O}(10^{-5}), \quad (B.24)$$

where θ is the typical angular separation of galaxy-galaxy gravitational lensing systems in radians. We will work in a regime where

$$Hr_{s,o} \sim \mathcal{O}(1), \quad Hr_c \sim r_c/r_{s,o} \sim \mathcal{O}(x); \quad (B.25)$$

since $r_{s,o}$ are connected to angular diameter distances $D_A(z_i, z_j)$ (as we show in App. C.4), this means that we are considering $D_A(z_i, z_j) \sim 1/H \sim \text{Gpc}$. We however recall that static coordinates are formally valid only for $r < 1/H$. Using

$$\frac{r_{Sc}}{|\tilde{b}|} = x \sqrt{(1 - x)}, \quad (B.26)$$

approximating

$$u = \sum_{n=0}^{\infty} u_n x^n, \quad (B.27)$$

and solving iteratively Eq. (B.23), we find

$$u_c = x, \quad (B.28)$$

$$u_1 = -x + x^2 - x^3 + 2x^4 + \mathcal{O}(x^5), \quad (B.29)$$

$$u_2 = 1 - x^2 + x^3 - 2x^4 + \mathcal{O}(x^5), \quad (B.30)$$

where for completeness we show all the three solutions together with the (inverse) closest approach distance u_c . We can then express ($p := u/u_c$; we stop at order x^3 ; recall $u_s \sim \mathcal{O}(x^2)$, $u_c \sim \mathcal{O}(x)$)

$$\begin{aligned} \int_{u_s}^{u_c} \frac{du}{\sqrt{x^2 - u^2(1 - u)}} &= \frac{1}{\sqrt{u_c}} \int_{u_s/u_c}^1 \frac{dp}{\sqrt{(1 - p)(p - u_1/u_c)(u_2/u_c - p)}} \\ &= \frac{\pi}{2} + x - \frac{u_s}{x} + \frac{15}{32} \pi x^2 - \frac{x^2}{2} - \frac{u_s}{2} - \frac{15}{32} \pi x^3 - \frac{3}{8} x u_s - \frac{1}{6} \frac{u_s^3}{x^3} + \mathcal{O}(x^4); \end{aligned} \quad (B.31)$$

we can hence write, from Eq. (B.12)

$$s(\varphi_o - \varphi_s + \pi) = 2\frac{r_{\text{Sc}}}{r_c} - \frac{r_s + r_{o1}}{r_s r_o} \left(r_c + \frac{r_{\text{Sc}}}{2} \right) + \left(\frac{15}{16}\pi - 1 \right) \frac{r_{\text{Sc}}^2}{r_c^2} + \mathcal{O}(x^3). \quad (\text{B.32})$$

For reference, the connection between $b, \tilde{b}, r_c$ when expanding on x , reads

$$|b| = |\tilde{b}| \left(1 + \frac{H^2 b^2}{2} + \mathcal{O}(x^4) \right) = r_c \left(1 + \frac{r_{\text{Sc}}}{2r_c} + \frac{3}{8} \frac{r_{\text{Sc}}^2}{r_c^2} + \frac{H^2 r_c^2}{2} + \mathcal{O}(x^3) \right). \quad (\text{B.33})$$

We want to connect Eq. (B.32) with the observed angle Eq. (B.14). To do that, we expand on peculiar velocities v_o as well. Assuming $v \sim x \sim 10^{-5}$ (even though peculiar velocities are expected to be somewhat larger, $v \sim 10^{-4}$), we can expand up to $\mathcal{O}(x^2)$ the relation Eq. (B.15) and obtain

$$E_o = 1 + Hr_o v_o + \frac{v_o^2}{2} + \mathcal{O}(x^3). \quad (\text{B.34})$$

In the previous, we are assuming v_o to be either positive or negative.

Before applying Eq. (B.14) to Eq. (B.32), notice that, since $Hr_c \sim x$, we have

$$s \frac{r}{|b|} = x \sqrt{1 - x - H^2 r_c^2} = x - \frac{x^2}{2} - \frac{x^3}{8} - x \frac{H^2 r_c^2}{2} + \mathcal{O}(x^4), \quad (\text{B.35})$$

which allows us to write

$$\theta = (1 + Hr_o) \frac{|\tilde{b}|}{r_{o1}} \left(1 + v_o + \frac{v_o^2}{2} \right) + \mathcal{O}(x^3). \quad (\text{B.36})$$

Using the relation

$$r_c = |\tilde{b}| - \frac{r_s}{2} - \frac{3r_s^2}{8|\tilde{b}|} + \mathcal{O}(x^3), \quad (\text{B.37})$$

and the results in App. C.4 (in particular the unlensed angular diameter distance expressions Eq. (C.78)⁵, we can write

$$\theta = (1 + Hr_o) \frac{|\tilde{b}|}{r_{o1}} + (1 + H^3 r_{o1}^3) \frac{|\tilde{b}|^3}{6r_{o1}^3} + \mathcal{O}(x^5), \quad (\text{B.38})$$

and thus rewrite Eq. (B.32) in the familiar form

$$s \frac{D_s}{D_{\text{ls}}} \beta = \frac{D_s}{D_{\text{ls}}} \theta - 2 \frac{r_{\text{Sc}}}{D_1 \theta} - \frac{15\pi}{16} \left(\frac{r_{\text{Sc}}}{D_1 \theta} \right)^2 + \mathcal{O}(x^3), \quad (\text{B.39})$$

where we defined $\beta := -(\varphi_o - \varphi_s + \pi)D_{\text{ls}}/D_s$. In Eq. (B.39), is it assumed that θ is positive, and the two images solutions come from $s = +1$ or $s = -1$ respectively. In Eq. (B.39), at order $\mathcal{O}(x^2)$, effects of H^2 are completely encoded in the (unlensed) angular diameter distances, hence we remark our agreement with Ref. [But16]. Non-trivial cosmological terms start appearing at $\mathcal{O}(x^3)$. Notice that the lowest order result of the previous agrees with Eq. (2.28) with $\alpha \propto 2r_{\text{Sc}}/(D_1\theta)$, valid for a Schwarzschild lens, Eq (2.40).

⁵Here, $D_{s,\text{ls}}$ will always be unlensed angular diameter distances, i.e. angular diameter distance as they would be defined in the background de-Sitter spacetime without lens. We discuss more on this in App. C.4.

Einstein angle. From (B.39), the Einstein angle corresponds to the case $\beta = 0$, resulting in

$$\theta_E = \sqrt{\frac{2r_{\text{Sc}}D_{\text{ls}}}{D_sD_l}} + \frac{15\pi}{64} \frac{r_{\text{Sc}}}{D_l} + \mathcal{O}(x^3), \quad (\text{B.40})$$

where we see that the leading order is the known Einstein angle result for a point mass/Schwarzschild lens. Again, the effect of cosmology up to $\mathcal{O}(x^2)$ is completely captured by the angular diameter distances, in contrast with e.g. the claims in [Pia16].

B.3 Redshift expressions in the Kottler metric

In the small angle and peculiar velocities approximation, the source redshift expression Eq. (B.18) expanded in powers of x yields

$$1 + z_s = \frac{1 + Hr_o}{1 - Hr_s} \left(1 + (v_s + v_o) + \frac{(v_s + v_o)^2}{2} \right) \left(1 - H \frac{r_s + r_o}{2r_s r_o} b^2 \right) + \frac{r_{\text{Sc}}}{2H} \left(\frac{1}{(1 + Hr_o)r_o^2} + \frac{1}{(1 - Hr_s)r_s^2} \right) + \mathcal{O}(x^3). \quad (\text{B.41})$$

It will also be useful to see how Eq. (B.18) reduces to familiar results for $r_{\text{Sc}} = 0$, $b = 0$ and $v_i = 0$; we have

$$1 + z_s = \frac{1 + Hr_o}{1 - Hr_s}. \quad (\text{B.42})$$

For the expression of the lens redshift Eq. (B.20), the discussion is more involved. Assuming one can somehow single out a light ray with $b = 0$ and corresponding to a circular orbit ($J(r) = 0$), coming from a source at distance $r_{\text{lens}} \sim r_c$, with $E := 1 + \Delta E_l$, $\Delta E_l \sim \mathcal{O}(x)$, Eq. (B.20) would read

$$1 + z_l = (1 + Hr_o) \left(1 + v_o + \frac{v_o^2}{2} \right) + \frac{r_{\text{Sc}}}{2Hr_o^2} + \mathcal{O} \left(\Delta E_l, \frac{r_{\text{Sc}}}{r_{\text{lens}}}, x^3 \right). \quad (\text{B.43})$$

More realistic lens redshift measurements would take an average over the lens light. Such an average is unlikely to yield a resulting redshift measurement which does not suffer $\mathcal{O}(x)$ corrections connected to how such procedure is performed. In the limit $r_{\text{Sc}} = 0$, there should be no problem in setting $r_{\text{lens}} = 0$, yielding the exact result

$$z_l = Hr_o. \quad (\text{B.44})$$

Connection with FLRW results. As a sanity check, we want now to connect the definitions of redshift Eq. (B.44), (B.42) with the standard FLRW result

$$1 + z_s = e^{H(T_{o1} - T_s)}, \quad (\text{B.45})$$

where T denotes the comoving time at emitter and at the observation events.

For a massive comoving observer (in the following, we ignore r_{Sc} terms which would involve $\mathcal{O}(x^2)$ terms)

$$\frac{dr}{dt} = \frac{\dot{r}}{\dot{t}} = \pm(1 - H^2 r^2) Hr, \quad (\text{B.46})$$

from which we can write

$$\frac{1}{H} \log \frac{r_{o1} \sqrt{1 - H^2 r_s^2}}{r_s \sqrt{1 - H^2 r_{o1}^2}} = t_{o1} - t_s \implies \frac{1}{H} \ln \frac{r_{o1}}{r_s} = t_{o1} - t_s + \frac{1}{2H} \ln \frac{1 - H^2 r_{o1}^2}{1 - H^2 r_s^2}; \quad (\text{B.47})$$

using

$$\frac{dT}{dr} = \frac{1}{Hr} \implies T_{o1} - T_s = \frac{1}{H} \ln \frac{r_{o1}}{r_s}, \quad (\text{B.48})$$

we can rewrite Eq. (B.47) as

$$T_s - T_{o1} = t_s - t_{o1} + \frac{1}{2H} \ln \frac{1 - H^2 r_s^2}{1 - H^2 r_{o1}^2}. \quad (\text{B.49})$$

To connect the previous with the redshift factor Eq. (B.42), we need to use expressions for a massless geodesic,

$$t_{o1} - t_s = \int_{-r_s}^{r_{o1}} \frac{dr}{1 - H^2 r^2} = \frac{1}{2H} \ln \frac{(1 + Hr_s)(1 + Hr_{o1})}{(1 - Hr_s)(1 - Hr_{o1})}. \quad (\text{B.50})$$

Now, using Eq. (B.50) to express $t_{o1} - t_s$ in Eq. (B.47), one indeed recovers

$$1 + z = e^{H(T_{o1} - T_s)} = \frac{1 + Hr_{o1}}{1 - Hr_s}. \quad (\text{B.51})$$

Eq. (B.44) follows straightforwardly by setting $r_s = 0$.

B.4 Time delays in the small angle approximation

We now focus on the time delays computation, Eqs. (B.17),(B.21),(B.22). In the small angle approximation, we can write

$$\tilde{b}^2 = \frac{r_c^3}{r_c - r_{Sc}} = \frac{r_c^2}{1 - x}, \quad \frac{\tilde{b}}{b} = \sqrt{1 - H^2 \tilde{b}^2} = \sqrt{1 - \frac{H^2 r_c^2}{1 - x}} = 1 - \frac{1}{2} H^2 r_c^2 - \frac{1}{2} H^2 r_c^2 x + \mathcal{O}(x^3); \quad (\text{B.52})$$

and conveniently express

$$\sqrt{1 - \frac{b^2}{r^2} A(r)} = \frac{b}{\tilde{b}} \sqrt{1 - \frac{r_c^2}{r^2(1 - x)} \left(1 - x \frac{r_c}{r}\right)}. \quad (\text{B.53})$$

Expanding Eq. (B.21) up to second order in $r_{Sc} = r_c x$, we have⁶

$$\begin{aligned} \int_{r_c}^{r_s} \frac{dr}{A(r) \sqrt{1 - b^2 A(r)/r^2}} &= \frac{\tilde{b}}{b} \int_{r_c}^{r_s} \frac{dr}{(1 - x r_c/r - H^2 r^2) \sqrt{1 - \frac{r_c^2}{r^2(1-x)} \left(1 - x \frac{r_c}{r}\right)}} \\ &\simeq \frac{\tilde{b}}{b} \int_{r_c}^{r_s} \left(\frac{r}{(1 - H^2 r^2) \sqrt{r^2 - r_c^2}} + \frac{r_c x (3r_c + r(2 - H^2 r r_c))}{2(r + r_c) \sqrt{r^2 - r_c^2} (1 - H^2 r^2)^2} + \mathcal{O}(r_s^2) \right) \\ &\simeq \frac{\tilde{b}}{b} \frac{1}{H \sqrt{1 - H^2 r_c^2}} \text{ArcTanh} \frac{H \sqrt{r^2 - r_c^2}}{\sqrt{1 - H^2 r_c^2}} \Big|_{r_c}^{r_s} + \frac{\tilde{b}}{b} \frac{r_c x}{2} \left(\frac{\sqrt{r - r_c} (1 + H^2 r r_c)}{\sqrt{r + r_c} (1 - H^2 r_c^2) (1 - H^2 r^2)} \right. \\ &\quad \left. + \frac{2 + H r_c - 2 H^2 r_c^2}{(1 - H^2 r_c^2)^{3/2}} \text{ArcTanh} \frac{\sqrt{(r - r_c)(1 + H r_c)}}{\sqrt{(r + r_c)(1 - H r_c)}} + \frac{2 - H r_c - 2 H^2 r_c^2}{(1 - H^2 r_c^2)^{3/2}} \text{ArcTanh} \frac{\sqrt{(r - r_c)(1 - H r_c)}}{\sqrt{(r + r_c)(1 + H r_c)}} \right) \Big|_{r_c}^{r_s} \\ &\quad + \mathcal{O}(r_{Sc}^2). \end{aligned} \quad (\text{B.54})$$

⁶The $\mathcal{O}(r_{Sc}^2)$ is involved, but part of it will contribute at level $r_{Sc} \mathcal{O}(x)$ to the time delay, hence it is needed. We use it in computations, but we avoid writing it explicitly until Eq. (B.56) excluded.

We can write the total result for the travel time of image i as (we now reinstate the i pedices which label the image)

$$\begin{aligned}
t_{oi} = & \frac{\tilde{b}_i}{b_i} \frac{1}{H\sqrt{1-H^2r_{ci}^2}} \text{ArcTanh} \frac{H\sqrt{r_e^2-r_{ci}^2}}{\sqrt{1-H^2r_{ci}^2}} + \frac{\tilde{b}_i}{b_i} \frac{r_{Sc}}{2} \left(\frac{\sqrt{r_e-r_{ci}}(1+H^2rr_{ci})}{\sqrt{r_e+r_{ci}}(1-H^2r_{ci}^2)(1-H^2r_e^2)} \right. \\
& + \frac{2+Hr_{ci}-2H^2r_{ci}^2}{(1-H^2r_{ci}^2)^{3/2}} \text{ArcTanh} \frac{\sqrt{(r_e-r_{ci})(1+Hr_{ci})}}{\sqrt{(r_e+r_{ci})(1-Hr_{ci})}} \\
& \left. + \frac{2-Hr_{ci}-2H^2r_{ci}^2}{(1-H^2r_{ci}^2)^{3/2}} \text{ArcTanh} \frac{\sqrt{(r_e-r_{ci})(1-Hr_{ci})}}{\sqrt{(r_e+r_{ci})(1+Hr_{ci})}} \right) + \mathcal{O}(r_s^2) + (r_e \leftrightarrow r_{oi}) ;
\end{aligned} \tag{B.55}$$

expanding in r_{ci} (reminding that $x \sim Hr_c \sim r_c/r_{s,o}$), the travel time reduces to

$$\begin{aligned}
t_{oi} = & \frac{1}{H} \text{ArcTanh} Hr_s + \frac{r_{Sc}}{2} \ln \frac{4r_s^2}{(1-H^2r_s^2)r_{ci}^2} + \frac{r_{Sc}}{2(1-H^2r_s^2)} - \frac{r_{Sc}r_{ci}}{2r_s} \\
& - \frac{r_{ci}^2}{2r_s} + \frac{r_{Sc}^2}{2r_{ci}} \left(\frac{15\pi}{8} - 1 \right) + r_{Sc} \mathcal{O}(x^2) + (r_s \leftrightarrow r_{oi}) ;
\end{aligned} \tag{B.56}$$

now, setting

$$y := \frac{r_{o2} - r_{o1}}{r_{o1}} , \tag{B.57}$$

we can see from Eq. (B.22) that y/H is of the same order of the time delay Δt , which cannot be order of $1/H$ (this would violate the assumption $r_{ci} \ll 1/H$), meaning that y is a small quantity we can expand over. Expanding over y Eq. (B.56), we end up with

$$\begin{aligned}
\Delta t = t_{o2} - t_{o1} = & 2r_{Sc} \ln \frac{r_{c1}}{r_{c2}} + \frac{1}{2} (r_{c1}^2 - r_{c2}^2) \left(\frac{1}{r_{o1}} + \frac{1}{r_s} \right) + \frac{r_{Sc}}{2} (r_{c1} - r_{c2}) \left(\frac{1}{r_s} + \frac{1}{r_{o1}} \right) \\
& + r_{Sc}^2 \left(\frac{1}{r_{c2}} - \frac{1}{r_{c1}} \right) \left(\frac{15\pi}{8} - 1 \right) + \frac{r_{o1}y}{1-H^2r_{o1}^2} \left(1 + \frac{H^2r_{o1}^2}{1-H^2r_{o1}^2} y \right) + r_{Sc} \mathcal{O}(x^2, y, yx) + r_{o1} \mathcal{O}(y^3) .
\end{aligned} \tag{B.58}$$

For later convenience, we define

$$\Delta t =: \Delta t' + \frac{r_{o1}y}{1-H^2r_{o1}^2} \left(1 + \frac{H^2r_{o1}^2}{1-H^2r_{o1}^2} y \right) + r_{Sc} \mathcal{O}(y, yx) + r_{o1} \mathcal{O}(y^3) , \tag{B.59}$$

that is, $\Delta t'$ is the y -independent part of Δt . Notice that $\Delta t' \sim \mathcal{O}(r_{Sc})$.

From Eq. (B.22), up to $r_{Sc} \mathcal{O}(y, yx)$, $r_{o1} \mathcal{O}(y^3)$,

$$\begin{aligned}
\Delta t \simeq & \frac{y}{H} \left(\frac{1}{1-H^2r_{o1}^2} \left(1 - \frac{1}{2} \frac{1-3H^2r_{o1}^2}{1-H^2r_{o1}^2} \left(y + \frac{r_{Sc}}{H^2r^3} \right) \right) - \frac{v_o}{Hr_o} \left(1 + \frac{v_o}{2Hr_o} \right) + \frac{3v_o^2}{2H^2r_{o1}^2} \right) \\
\simeq & \Delta t' + \frac{r_{o1}y}{1-H^2r_{o1}^2} \left(1 + \frac{H^2r_{o1}^2}{1-H^2r_{o1}^2} y \right) ;
\end{aligned} \tag{B.60}$$

from the previous equality, of the form $H\Delta t' = y \sum_{n=0} f_n y^n$, we see that $y \sim \mathcal{O}(H\Delta t')$, which implies $y \sim \mathcal{O}(x^2)^7$, so that

$$y = H\Delta t'(1 + Hr_{o1}) \left(1 + v_o + \frac{v_o}{Hr_{o1}} \right) + \mathcal{O}(x^4); \quad (\text{B.63})$$

from Eq. (B.17)

$$\Delta T = \frac{y}{H} \left(1 - \frac{v_o}{Hr_{o1}} \right) + r_{sc} \mathcal{O}(x^2), \quad (\text{B.64})$$

which implies

$$\Delta T = (1 + Hr_{o1})(1 + v_o)\Delta t' + r_{sc} \mathcal{O}(x^2), \quad (\text{B.65})$$

ending up with

$$\begin{aligned} \Delta T = (1 + Hr_{o1})(1 + v_o) & \left(2r_{sc} \ln \frac{r_{c1}}{r_{c2}} + \frac{1}{2}(r_{c1}^2 - r_{c2}^2) \left(\frac{1}{r_{o1}} + \frac{1}{r_s} \right) + \frac{r_{sc}}{2}(r_{c1} - r_{c2}) \left(\frac{1}{r_s} + \frac{1}{r_{o1}} \right) \right. \\ & \left. + r_{sc}^2 \left(\frac{1}{r_{c2}} - \frac{1}{r_{c1}} \right) \left(\frac{15\pi}{8} - 1 \right) \right) + r_{sc} \mathcal{O}(x^2). \end{aligned} \quad (\text{B.66})$$

Using

$$\theta_i = (1 + z_l) \left(\frac{|\tilde{b}|}{r_{oi}} + \mathcal{O}(x^3) \right) \implies r_{ci} = D_l \theta_i - \frac{r_{sc}}{2} + r_s \mathcal{O}(x^3), \quad \frac{1}{r_{o1}} + \frac{1}{r_s} = \frac{D_s}{D_l D_{ls}}, \quad (\text{B.67})$$

together with Eq. (B.20) to express $(1 + Hr_{o1})(1 + v_o)$ as a function of the measured lens redshift, we can finally write

$$\Delta T = (1 + z_l) \left(2r_{sc} \ln \frac{\theta_1}{\theta_2} + \frac{1}{2} \frac{D_s D_l}{D_{ls}} (\theta_1^2 - \theta_2^2) + \frac{15\pi r_{sc}^2}{8 D_l} \left(\frac{1}{\theta_2} - \frac{1}{\theta_1} \right) \right) + r_{sc} \mathcal{O}(x^2). \quad (\text{B.68})$$

Fermat principle. According to Fermat principle, the light travel time is stationary with respect to perturbations *which hold the source position fixed* (for a precise statement, see e.g. [SEF92]). Taking the $\vec{\theta}$ derivative of Eq. (B.68) would not yield Eq. (B.39), because one has to ensure that the source position β remains constant. One can impose this using the Lagrange multiplier γ , via the auxiliary function

$$\tilde{T}(\theta, \gamma) = T(\theta) + \gamma \left(\frac{D_s}{D_{ls}} \beta - sf(\theta) \right), \quad (\text{B.69})$$

where $T(\theta)$ is the total travel time of the light ray passing at θ as seen by the observer and $f(\theta) := \frac{D_s}{D_{ls}} \theta - 2 \frac{r_{sc}}{D_l \theta} - \frac{15\pi}{16} \left(\frac{r_{sc}}{D_l \theta} \right)^2$ from Eq. (B.39). The stationary points of $\tilde{T}$ with respect to θ and γ are also the stationary points of $T(\theta)$ with the constraint $\frac{D_s}{D_{ls}} \beta = sf(\theta)$. Then,

$$\frac{\partial T}{\partial \theta} - \gamma \frac{\partial f}{\partial \theta} = 0 \implies \gamma = -s\theta D_l; \quad (\text{B.70})$$

⁷ As far as peculiar velocities are concerned, notice that

$$E_o = 1 + Hr_{o1}v_{o1} + \frac{v_{o1}^2}{2} + \mathcal{O}(x^3) = 1 + Hr_{o2}v_{o2} + \frac{v_{o2}^2}{2} + \mathcal{O}(x^3); \quad (\text{B.61})$$

For this relation to hold at lowest order, writing $v_{o2} = v_{o1}(1 + \Delta v)$ and $r_{o2} = r_{o1}(1 + y)$, with $y \sim \mathcal{O}(x^2)$,

$$0 = Hr_{o1}v_{o1}(y + \Delta v + y\Delta v) + \mathcal{O}(x^3) \implies \Delta v = -y + \mathcal{O}(x^3), \quad (\text{B.62})$$

meaning that differences of peculiar velocities between the two images are order $v_{o1}\Delta v \sim \mathcal{O}(x^3)$, negligible at the order we are interested.

notice that $\tilde{T} = T$ (we are adding and subtracting the same quantity), but now we can write the time delay as

$$\begin{aligned} \Delta\tilde{T} = \Delta T = (1+z_1) & \left(\frac{1}{2} \frac{D_s D_1}{D_{ls}} (\theta_2^2 - \theta_1^2) - \frac{D_s D_1}{D_{ls}} \beta (s_2 \theta_2 - s_1 \theta_1) - 2r_{sc} \ln \frac{\theta_2}{\theta_1} \right. \\ & \left. + \frac{15\pi r_{sc}^2}{16D_1} \left(\frac{1}{\theta_2} - \frac{1}{\theta_1} \right) \right) + r_{sc} \mathcal{O}(x^2), \end{aligned} \quad (\text{B.71})$$

whose gradient clearly gives back Eq. (B.39). The fact that γ can depend on θ does not allow one to recover the expression for the time delays via integrating the angle equation, Eq. (B.39).

Eq. (B.71) is our final result. Notice that, at this order, cosmology enters only via (unlensed) angular diameter distances. Non-trivial cosmological effect start entering at $r_{sc} \mathcal{O}(x^2)$, arguably too small to be observed. Notice also that the zeroth order of Eq. (B.71) is the usual time delay formula Eq. (2.49), with $\Psi(\theta) \propto 2r_{sc} \ln \theta$ for a Schwarzschild lens, Eq. (2.40).

Appendix C

Some General Relativity Concepts and Derivations

We here show some basic concepts and derivations of GR which were used in the main text, together with an account of angular diameter distance in a generic metric, App. C.4.

C.1 Some basic definitions

Sign conventions. Throughout this thesis, we use the following sign conventions:

- Metric: [S1](+ - - -);
- Christoffel symbols

$$\Gamma_{\mu\nu}^{\rho} = \frac{1}{2}g^{\rho\sigma}(\partial_{\mu}g_{\nu\sigma} + \partial_{\nu}g_{\mu\sigma} - \partial_{\sigma}g_{\mu\nu}) ; \quad (C.1)$$

- Riemann tensor:

$$R^{\mu}_{\alpha\beta\gamma} = 2[S2](\partial_{[\beta}\Gamma^{\mu}_{\gamma]\alpha} + \Gamma^{\mu}_{\sigma[\beta}\Gamma^{\sigma}_{\gamma]\alpha}) ; \quad (C.2)$$

- Ricci tensor:

$$R_{\mu\nu} = [S2][S3]R^{\alpha}_{\mu\alpha\nu} ; \quad (C.3)$$

- Einstein equations:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = [S3]8\pi GT_{\mu\nu} , \quad (C.4)$$

with [S1] = [S2] = [S3] = 1. $g_{\mu\nu}$ are the metric coefficients, R is the Ricci scalar, $R = g^{\mu\nu}R_{\mu\nu}$, and $T_{\mu\nu}$ is the energy-momentum tensor.

Covariant derivative. The covariant derivative ∇_{μ} of a tensor $T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p}$ reads

$$\nabla_{\mu}T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} = \partial_{\mu}T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\nu_p} + \Gamma^{\mu_1}_{\mu\sigma}T^{\sigma\cdots\mu_p}_{\nu_1\cdots\nu_p} + \cdots + \Gamma^{\mu_p}_{\mu\sigma}T^{\mu_1\cdots\sigma}_{\nu_1\cdots\nu_p} - \Gamma^{\sigma}_{\mu\nu_1}T^{\mu_1\cdots\mu_p}_{\sigma\cdots\nu_p} - \cdots - \Gamma^{\sigma}_{\mu\nu_p}T^{\mu_1\cdots\mu_p}_{\nu_1\cdots\sigma} . \quad (C.5)$$

Weyl tensor. The Weyl tensor $C_{\mu\nu\rho\sigma}$, which is the trace-free part of the Riemann tensor, can be expressed as

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} + \frac{1}{2}(R_{\mu\sigma}g_{\nu\rho} - R_{\mu\rho}g_{\nu\sigma} + R_{\nu\rho}g_{\mu\sigma} - R_{\nu\sigma}g_{\mu\rho}) + \frac{R}{6}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) . \quad (C.6)$$

C.2 Scalar fields in GR

The full action for a scalar field ϕ minimally coupled to gravity reads

$$S_{\text{matter}} = \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right) =: \int d^4x \sqrt{-g} \left(\frac{R}{16\pi G} + \mathcal{L} \right); \quad (\text{C.7})$$

g is the determinant of the metric, R is the Ricci scalar, which enters the so-called Einstein-Hilbert action

$$S_{\text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R. \quad (\text{C.8})$$

The relation of R with the metric coefficients $g_{\mu\nu}$ can be read in Eqs. (C.1),(C.2),(C.3).

The equation of motion for ϕ follows from

$$\frac{\delta S}{\delta \phi} = 0 \implies \frac{\partial \sqrt{-g} \mathcal{L}}{\partial \phi} = \partial_\mu \frac{\partial \sqrt{-g} \mathcal{L}}{\partial \partial_\mu \phi}, \quad (\text{C.9})$$

which yields

$$-\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} (\sqrt{-g} g^{\mu\nu} \partial_\nu \phi) + \frac{\partial V}{\partial \phi} = 0. \quad (\text{C.10})$$

Einstein equations are obtained via

$$\frac{\delta S}{\delta g^{\mu\nu}} = 0 \implies R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G \left(-\frac{2}{\sqrt{-g}} \frac{\partial \sqrt{-g} \mathcal{L}}{\partial g^{\mu\nu}} \right). \quad (\text{C.11})$$

the previous is exactly the Einstein equation, granted one identifies

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}. \quad (\text{C.12})$$

The energy-momentum tensor, exploiting

$$\frac{\partial \sqrt{-g}}{\partial g^{\mu\nu}} = -\frac{\sqrt{-g} g_{\mu\nu}}{2}, \quad (\text{C.13})$$

reads

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left(\frac{1}{2} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi + V(\phi) \right), \quad (\text{C.14})$$

which can be recast in the form of a perfect fluid one with the identifications

$$\rho = -\frac{1}{2} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi + V(\phi), \quad P = -\frac{1}{2} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi - V(\phi), \quad u_\mu = \frac{1}{\sqrt{-g^{\rho\nu} \partial_\rho \phi \partial_\nu \phi}} \partial_\mu \phi. \quad (\text{C.15})$$

C.3 Deriving Friedmann and perturbed FRW Einstein equations

We here derive the Einstein equations for a ULDM metric. Eq. (1.5), which will result in the Friedmann equations, Eq. (1.11), (1.12), (1.15). In Sec. C.3.1, we will instead derive the Einstein equations for a perturbed flat FLRW metric in the Newtonian gauge, relevant for Sec. 3.1.

We will use the tetrad (Vierbein) formalism, which employs the tetrad $e^{\hat{\mu}}$ to rewrite the metric as

$$ds^2 = \eta_{\hat{\mu}\hat{\nu}} e^{\hat{\mu}} e^{\hat{\nu}}, \quad (C.16)$$

where $\eta_{\hat{\mu}\hat{\nu}}$ is the Minkowski metric, and

$$e^{\hat{0}} = a d\eta =: e^{\hat{0}}_0 d\eta, \quad e^{\hat{i}} = e^{\hat{i}}_i dx^i, \quad e^{\hat{j}} = (a, a f_\kappa, a f_\kappa \sin \theta) \delta^{\hat{j}}_j, \quad x^i = (w, \theta, \varphi); \quad (C.17)$$

to avoid confusion, hatted indices will always refer to tetrad indices. The idea of the tetrad formalism is to work in a frame where the metric looks locally Minkowskian (thanks to the equivalence principle, this is always possible). We will denote the inverse of $e^{\hat{\mu}}_\nu$ with $e^{\hat{\nu}}_\rho$ (such that $e^{\hat{\mu}}_\nu e^{\hat{\nu}}_\rho = \delta^{\hat{\mu}}_\rho$ and $e^{\hat{\nu}}_\mu e^{\hat{\mu}}_\rho = \delta^{\hat{\nu}}_\rho$). We will need the so-called spin connections $\omega^{\hat{\mu}}_{\hat{\nu}}$, which can be found with¹

$$de^{\hat{\mu}} =: -\omega^{\hat{\mu}}_{\hat{\nu}} \wedge e^{\hat{\nu}}, \quad (C.20)$$

implying (we denote with ' derivatives with respect to the argument of the respective functions)

$$\begin{aligned} de^{\hat{0}} &= -\omega^{\hat{0}}_{\hat{j}} \wedge e^{\hat{j}} = 0, \\ de^{\hat{1}} &= -\omega^{\hat{1}}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a' d\eta \wedge dw; \\ de^{\hat{2}} &= -\omega^{\hat{2}}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a' f_\kappa d\eta \wedge d\theta + a f'_\kappa dw \wedge d\theta; \\ de^{\hat{3}} &= -\omega^{\hat{3}}_{\hat{\mu}} \wedge e^{\hat{\mu}} = a' f_\kappa \sin \theta d\eta \wedge d\varphi + a f'_\kappa \sin \theta dw \wedge d\varphi + a f_\kappa \cos \theta d\theta \wedge d\varphi. \end{aligned} \quad (C.21)$$

From the previous, we can read (we can lower or raise hatted indices with the Minkowski metric, $\omega_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\rho}} \omega^{\hat{\rho}}_{\hat{\nu}}$ etc., and notice $\omega_{\hat{\mu}\hat{\nu}} = -\omega_{\hat{\nu}\hat{\mu}}$)

$$\begin{aligned} \omega^{\hat{0}}_{\hat{1}} &= \frac{a'}{a} dw = \omega^{\hat{1}}_{\hat{0}}, \\ \omega^{\hat{0}}_{\hat{2}} &= \frac{a'}{a} f_\kappa d\theta = \omega^{\hat{2}}_{\hat{0}}, \\ \omega^{\hat{0}}_{\hat{3}} &= \frac{a'}{a} f_\kappa \sin \theta d\varphi = \omega^{\hat{3}}_{\hat{0}}, \\ \omega^{\hat{1}}_{\hat{2}} &= -f'_\kappa d\theta = -\omega^{\hat{2}}_{\hat{1}}, \\ \omega^{\hat{1}}_{\hat{3}} &= -f'_\kappa \sin \theta d\varphi = -\omega^{\hat{3}}_{\hat{1}}, \\ \omega^{\hat{2}}_{\hat{3}} &= -\cos \theta d\varphi = -\omega^{\hat{3}}_{\hat{2}}. \end{aligned} \quad (C.22)$$

¹Recall the formula for exterior derivative

$$d(a_{\mu\dots\sigma} dx^\mu \wedge \dots \wedge dx^\sigma) = (\partial_\nu a_{\mu\dots\sigma}) dx^\nu \wedge dx^\mu \wedge \dots \wedge dx^\sigma, \quad (C.18)$$

where the outer product of differentials is defined as

$$dx^\mu \wedge dx^\nu = \frac{1}{2}(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu), \quad (C.19)$$

where $\otimes$ denotes the tensor product.

Once we have the spin connection ω , the Riemann tensor can be found exploiting²

$$R^{\hat{\mu}}_{\hat{\nu}} := d\omega^{\hat{\mu}}_{\hat{\nu}} + \omega^{\hat{\mu}}_{\hat{\sigma}} \wedge \omega^{\hat{\sigma}}_{\hat{\nu}} = \frac{1}{2} e^{\hat{\mu}}_{\rho} e^{\sigma}_{\hat{\nu}} R^{\rho}_{\sigma\alpha\beta} dx^{\alpha} \wedge dx^{\beta} ; \quad (\text{C.24})$$

We need ($\mathcal{H} := a'/a$)

$$\begin{aligned} R^{\hat{0}}_{\hat{1}} &= \mathcal{H}' d\eta \wedge d\omega ; \\ R^{\hat{0}}_{\hat{2}} &= \mathcal{H}' f_{\kappa} d\eta \wedge d\theta , \\ R^{\hat{0}}_{\hat{3}} &= \mathcal{H}' f_{\kappa} \sin \theta d\varphi \wedge d\theta , \\ R^{\hat{1}}_{\hat{2}} &= (\mathcal{H}^2 f_{\kappa} - f_{\kappa}'') dw \wedge d\theta , \\ R^{\hat{1}}_{\hat{3}} &= (\mathcal{H}^2 f_{\kappa} - f_{\kappa}'') \sin \theta dw \wedge d\varphi , \\ R^{\hat{2}}_{\hat{3}} &= (\mathcal{H}^2 f_{\kappa}^2 - f_{\kappa}'^2 + 1) \sin \theta d\theta \wedge d\varphi . \end{aligned} \quad (\text{C.25})$$

From this, one recovers (we remark that sum over repeated indices is not implied in the following equations)

$$\begin{aligned} e^{\hat{0}}_0 e^{\hat{1}}_1 R^0_{1\mu\nu} &= \frac{a}{a'} R^0_{1\mu\nu} \implies R^0_{101} = \mathcal{H}' , \\ e^{\hat{0}}_0 e^{\hat{2}}_2 R^0_{2\mu\nu} &= \frac{1}{f_{\kappa}} R^0_{2\mu\nu} \implies R^0_{202} = \mathcal{H}' f_{\kappa}^2 , \\ e^{\hat{0}}_0 e^{\hat{3}}_3 R^0_{3\mu\nu} &= \frac{1}{f_{\kappa} \sin \theta} R^0_{3\mu\nu} \implies R^0_{303} = \mathcal{H}' f_{\kappa}^2 \sin^2 \theta , \\ e^{\hat{1}}_1 e^{\hat{2}}_2 R^1_{2\mu\nu} &= \frac{1}{f_{\kappa}} R^1_{2\mu\nu} \implies R^1_{212} = (\mathcal{H}^2 f_{\kappa} - f_{\kappa}'') f_{\kappa} , \\ e^{\hat{1}}_1 e^{\hat{3}}_3 R^1_{3\mu\nu} &= \frac{1}{f_{\kappa} \sin \theta} R^1_{3\mu\nu} \implies R^1_{313} = (\mathcal{H}^2 f_{\kappa} - f_{\kappa}'') f_{\kappa} \sin^2 \theta , \\ e^{\hat{2}}_2 e^{\hat{3}}_3 R^2_{3\mu\nu} &= \frac{1}{\sin \theta} R^2_{3\mu\nu} \implies R^2_{323} = (\mathcal{H}^2 f_{\kappa}^2 - f_{\kappa}'^2 + 1) \sin^2 \theta . \end{aligned} \quad (\text{C.26})$$

Ricci tensor reads (use relations like $R^{\mu}_{\nu\mu\nu} = g^{\mu\rho} g_{\nu\sigma} R^{\sigma}_{\rho\nu\mu}$)

$$\begin{aligned} R_{00} &= R^i_{0i0} = -R^0_{101} - \frac{1}{f_{\kappa}^2} R^0_{202} - \frac{1}{f_{\kappa}^2 \sin^2 \theta} R^0_{303} = -3\mathcal{H}' , \\ R_{11} &= \mathcal{H}' + 2\mathcal{H}^2 - 2\frac{f_{\kappa}''}{f_{\kappa}} , \\ R_{22} &= \mathcal{H}' f_{\kappa}^2 + 2\mathcal{H}^2 f_{\kappa}^2 - f_{\kappa} f_{\kappa}'' - f_{\kappa}'^2 + 1 , \\ R_{33} &= (\mathcal{H}' f_{\kappa}^2 + 2\mathcal{H}^2 f_{\kappa}^2 - f_{\kappa} f_{\kappa}'' - f_{\kappa}'^2 + 1) \sin^2 \theta ; \end{aligned} \quad (\text{C.27})$$

and the Ricci scalar

$$a^2 R = a^2 g^{\mu\nu} R_{\mu\nu} = -6\mathcal{H}' - 6\mathcal{H}^2 + 4\frac{f_{\kappa}''}{f_{\kappa}} - \frac{2}{f_{\kappa}^2} (1 - f_{\kappa}'^2) = -6\mathcal{H}' - 6\mathcal{H}^2 - 6\kappa , \quad (\text{C.28})$$

where on the last step we used the explicit form of f_{κ} for all possible κ .

²For reference, we show the Christoffel symbols formula in the tetrad formalism,

$$\Gamma^{\sigma}_{\mu\rho} = e^{\sigma}_{\hat{a}} \omega^{\hat{a}}_{\hat{b}\mu} e^{\hat{b}}_{\rho} + e^{\sigma}_{\hat{a}} \partial_{\mu} e^{\hat{a}}_{\rho} , \quad (\text{C.23})$$

When using Einstein equations, one should use the stress energy tensor in the new coordinates; in particular, notice that $u^\mu = (1/a, 0, 0, 0)$, hence $T_{\mu\nu} = a^2 \text{diag}(\bar{\rho}, \bar{P}, \bar{P}, \bar{P})$. The first Einstein equation reads (we convert to comoving time t instead of conformal time η , so $\dot{} := \partial/\partial t$, $\mathcal{H} = aH$, $H = \dot{a}/a$)

$$R_{00} - \frac{1}{2}a^2 R = 3a^2 H^2 + 3\kappa = 8\pi G a^2 \bar{\rho} + \Lambda a^2 \implies H^2 = \frac{8\pi G}{3} \bar{\rho} + \frac{\Lambda}{3} - \frac{\kappa}{a^2}. \quad (\text{C.29})$$

The 11 Einstein equation instead reads

$$R_{11} + \frac{1}{2}a^2 R = a^2 \left(\frac{-2\ddot{a}}{a} - H^2 \right) - \kappa = 8\pi G a^2 \bar{P} - \Lambda a^2, \quad (\text{C.30})$$

ending with the second Friedmann equation (using the first to substitute for H^2)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{P}) + \frac{\Lambda}{3}. \quad (\text{C.31})$$

The third Friedmann equation can be obtained combining the first two.

C.3.1 Perturbed flat FRW in Newtonian gauge

In the Newtonian gauge, the perturbed flat FLRW metric reads

$$ds^2 = -(1 + 2\Phi) dt^2 + (1 - 2\Psi) a^2(t) dx^i dx^i. \quad (\text{C.32})$$

Here Φ and Ψ are the two gravitational fields. Ψ will appear only in this section, so confusion with the lensing potential $\Psi(\vec{\theta})$ of the main text should not arise. Since we are mainly interested in the weak-field limit, to be applied in Sec. 3.1, in this derivation we allow ourselves to always ignore $H\Psi$, $H\Phi$ terms.

The vierbein read

$$e^{\hat{0}} = (1 + \Phi) dt =: e^{\hat{0}}_0 dt, \quad e^i = a(1 - \Psi) dx^i =: e^{\hat{i}}_i dx^i; \quad (\text{C.33})$$

the spin connections can be found with

$$\begin{aligned} de^{\hat{0}} &= \partial_i \Phi dx^i \wedge dt = -\omega^{\hat{0}}_{\hat{j}} \wedge e^{\hat{j}}, \\ de^{\hat{i}} &= a(H - \dot{\Psi}) dt \wedge dx^i - a\partial_j \Psi dx^j \wedge dx^i = -\omega^{\hat{i}}_{\hat{0}} \wedge e^{\hat{0}} - \omega^{\hat{i}}_{\hat{j}} \wedge e^{\hat{j}}, \end{aligned} \quad (\text{C.34})$$

which yield ($\dot{} = \partial_0$)

$$\begin{aligned} \omega^{\hat{0}}_{\hat{i}} &= \frac{\partial_i \Phi}{a} dt + a(H - \dot{\Psi}) dx^i = \omega^{\hat{i}}_{\hat{0}}, \\ \omega^{\hat{i}}_{\hat{j}} &= -\partial_j \Psi dx^i + \partial_i \Psi dx^j = -\omega^{\hat{j}}_{\hat{i}}. \end{aligned} \quad (\text{C.35})$$

The Riemann tensor can be found from

$$\begin{aligned} R^{\hat{0}}_{\hat{i}} &= \frac{\partial_i \partial_j \Phi}{a} dx^j \wedge dt + \partial_0(a(H - \dot{\Psi})) dt \wedge dx^i - a\partial_j \dot{\Psi} dx^j \wedge dx^i \\ &= \left(\frac{\partial_i \partial_j \Phi}{a} - a(H^2 + \dot{H} - \ddot{\Psi}) \delta_{ij} \right) dx^j \wedge dt - a\partial_j \dot{\Psi} dx^j \wedge dx^i; \\ R^{\hat{i}}_{\hat{j}} &= -\partial_j \dot{\Psi} dt \wedge dx^i + \partial_i \dot{\Psi} dt \wedge dx^j - \partial_k \partial_j \Psi dx^k \wedge dx^i + \partial_k \partial_i \Psi dx^k \wedge dx^j + a^2 H^2 dx^i \wedge dx^j. \end{aligned} \quad (\text{C.36})$$

One then recovers

$$\begin{aligned}
e_0^{\hat{0}} e_i^{\hat{i}} R^0_{i0j} &= \frac{1}{a} R^0_{i0j} = -\frac{\partial_i \partial_j \Phi}{a} + a(H^2 + \dot{H} - \ddot{\Psi}) \delta_{ij} , \\
e_0^{\hat{0}} e_i^{\hat{i}} R^0_{ijk} &= \frac{1}{a} R^0_{ijk} = -a \partial_j \dot{\Psi} \delta_{ik} + a \partial_k \dot{\Psi} \delta_{ij} , \\
e_i^{\hat{i}} e_j^{\hat{j}} R^i_{j0l} &= R^i_{j0l} = -\partial_j \dot{\Psi} \delta_{il} + \partial_i \dot{\Psi} \delta_{jl} , \\
e_i^{\hat{i}} e_j^{\hat{j}} R^i_{jkl} &= R^i_{jkl} = -\partial_k \partial_j \Psi \delta_{il} + \partial_k \partial_i \Psi \delta_{jl} + a^2 H^2 \delta_{ik} \delta_{jl} - (k \leftrightarrow l) .
\end{aligned} \tag{C.37}$$

The Ricci tensor reads

$$\begin{aligned}
R_{00} &= R^i_{i00} = -\frac{1}{a^2} R^0_{i0i} = \frac{\nabla^2 \Phi}{a^2} - 3(\dot{H} + H^2 - \ddot{\Psi}) , \\
R_{ij} &= R^0_{i0j} + R^k_{ikj} = -\partial_i \partial_j \Phi + a^2(\dot{H} + H^2 - \ddot{\Psi}) \delta_{ij} + \partial_i \partial_j \Psi + \delta_{ij} \nabla^2 \Psi + 2a^2 H^2 \delta_{ij} ,
\end{aligned} \tag{C.38}$$

and the Ricci scalar reads

$$R = g^{\mu\nu} R_{\mu\nu} = R_{00} - \frac{1}{a^2} R_{ii} = 2 \frac{\nabla^2 \Phi}{a^2} - 6(\dot{H} - \ddot{\Psi}) - 4 \frac{\nabla^2 \Psi}{a^2} - 12H^2 . \tag{C.39}$$

Finally, the first Einstein equation reads

$$R_{00} - \frac{1}{2} R = 3H^2 + 2 \frac{\nabla^2 \Psi}{a^2} = 8\pi G(\bar{T}_{00} + \delta T_{00}) . \tag{C.40}$$

$\bar{T}$ is the part of the energy momentum tensor which sources the background, cosmological evolution, and δT_{00} represents the fluctuations around this background. $\bar{T}_{00}$ is the source of the $3H^2$ term; from that, one recovers the first Friedmann equation (with $\kappa = 0$). For the fluctuation, we can write

$$\frac{\nabla^2 \Psi}{a^2} = 4\pi G \delta T_{00} . \tag{C.41}$$

The ij Einstein equation instead reads

$$R_{ij} + \frac{1}{2} \delta_{ij} a^2 R = \partial_i \partial_j (\Psi - \Phi) + a^2 \left(-2\dot{H} - 3H^2 + 2\ddot{\Psi} - \frac{\nabla^2 \Psi}{a^2} + \frac{\nabla^2 \Phi}{a^2} \right) \delta_{ij} = 8\pi G(\bar{T}_{ij} + \delta T_{ij}) . \tag{C.42}$$

Note that, from the background evolution, we obtain the second Friedmann equation,

$$a^2(-2\dot{H} - 3H^2) \delta_{ij} = a^2 \left(-2 \frac{\ddot{a}}{a} - H^2 \right) = 8\pi G g_{ik} g_{jl} g^{kl} \bar{p} = 8\pi G a^2 \delta_{ij} \bar{p} \implies \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\bar{\rho} + 3\bar{p}) . \tag{C.43}$$

For a perfect fluid, the traceless part of its energy-momentum tensor is zero, since we do not have anisotropic stresses in it. Coincidentally, the energy-momentum tensor of ULDM is a perfect fluid one, see Eq. (C.14), (C.15). Hence, taking the traceless part of Eq. (C.42) (meaning subtract $\delta_{ij} \text{tr}(\text{LHS})/3$ to the LHS, and analogous for the RHS), we have

$$\left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) (\Phi - \Psi) = 0 , \tag{C.44}$$

which indeed implies $\Phi = \Psi$ (in Fourier space, the equation reads $(k_i k_j - \frac{1}{3} k^2 \delta_{ij})(\Phi - \Psi)$; for $k \neq 0$, $\Phi_k = \Psi_k$; the case $k = 0$, assuming the appropriate decay at infinity, will also imply $\Phi_0 = \Psi_0$). For

reference, the trace part of Eq. (C.42), with $\Phi = \Psi$, reads

$$a^2 \ddot{\Phi} = \frac{8\pi G}{3} \text{tr}(\delta T_{ij}) . \quad (\text{C.45})$$

C.4 Angular diameter distance in a generic metric

In Sec. 1.1.2, we introduced a definition of angular diameter distance in FLRW metric. A rigorous definition of angular diameter distance, valid for any metric, reads instead [DR74; SEF92; Fle14]

$$\tilde{D}_A := \sqrt{\frac{dA_s}{d\Omega_o}} , \quad (\text{C.46})$$

where dA_s is the cross-section area of a light-ray bundle from the source and $d\Omega_o$ is the solid angle which subtends dA_s , as seen by the observer. We use the tilde symbol to differentiate this angular diameter distance, which in this context we can call lensed angular diameter distance, with the unlensed angular diameter distance D_A (i.e. the angular diameter distance which takes into account only the background FLRW metric) of the main text. Lensed angular diameter distances could be in principle observable, if one knows the true transverse area dA_s . However, for the lensing applications we discussed in this thesis, this is never really true, hence one would express gravitational lensing equations using just unlensed angular diameter distances, to be connected with observed redshifts. We anyway perform the calculation of $\tilde{D}_A$ in this appendix as an interesting exercise.

For obtaining an expression for $\tilde{D}_A$, we need a way to express dA_s . For this, we need to exploit the concept of ray bundles $x^\mu(\lambda, \vec{b})$, introduced in Sec. 2.1. In a general setting, $\vec{b}$ would label three real parameters, e.g. the two angles on the celestial sphere at which the light ray is received at the observer and the phase of the ray [SEF92]. However, for our purposes we can consider a one-dimensional ray bundle, with all the rays lying on the $\theta = \text{const}$ plane. For our case of interest, b can really be taken to be the b parameter related to r_c (or equivalently to φ_s), as introduced in App. B. Then, $\omega^\mu := dx^\mu/d\lambda$ is the geodesic vector and $\xi^\mu := dx^\mu/db$ is the geodesic deviation vector (connected to the distance between neighbouring light rays separated by δb). We assume the bundle to end at the observer point, so that $\xi(\lambda = \lambda_o) = 0$. A property we will need in the following is

$$\frac{d}{d\lambda}(\omega^\mu \xi_\mu) = \xi_\mu \underbrace{\omega^\nu \nabla_\nu \omega^\mu}_{=0} + \omega^\mu \underbrace{\frac{d\xi_\mu}{d\lambda}}_{=\frac{d\omega_\mu}{db}} = \frac{1}{2} \frac{d}{db}(\omega^\mu \omega_\mu) = 0 \implies \omega^\mu \dot{\xi}_\mu = 0 , \quad (\text{C.47})$$

where we exploited that ω^μ is a null geodesic vector. Using the Sachs formalism [JES61; Sac62], given an observer u^μ , we can define screen vectors s_A^μ , $A = 1, 2$, which projects on the screen perpendicular to both u^μ and ω^μ , similarly as we discussed in Sec. 2.1. We here remind the properties

$$g_{\mu\nu} s_A^\mu s_B^\nu = \delta_{AB} , \quad u_\mu s_A^\mu = \omega_\mu s_A^\mu = 0 , \quad \xi_A := s_A^\mu \xi_\mu . \quad (\text{C.48})$$

ξ_A obey Eq. (2.6), which is linear. This fact allows us to generically write

$$\xi_A(x_s) = C_{AB} \xi_B(x_o) + D_{AB} \frac{\dot{\xi}_B(x_o)}{\omega_o} ; \quad (\text{C.49})$$

the D_{AB} factor is the one we are interested into, since we assumed $\xi(x_o) = 0$; we will soon see that the observed angle $\theta \sim \dot{\xi}/\omega_o$. Hence

$$\tilde{D}_A = \sqrt{|\det(D_{AB})|} . \quad (\text{C.50})$$

In our setting, we have spherical symmetry, in particular we set no motion on θ , meaning we can write one of the screen vector to simply be

$$s_2^\mu = \left(0, 0, \frac{1}{r}, 0\right); \quad (\text{C.51})$$

if emitter, observer and the singularity at $r = 0$ are all aligned, then the only relevant screen vector would be s_1^μ , thus we can write a one-dimensional version of Eq. (C.46)

$$\tilde{D}_A = \left| \frac{dl_s(\lambda_s)}{d\theta_o} \right|, \quad (\text{C.52})$$

where θ_o is the angle as seen by the observer, and $dl_s = \xi_1(\lambda_s) db$ is the transverse distance of the two neighbouring light rays. With the notation of Eq. (C.50), we are saying that $D_{11} = D_{22} = dl_s(\lambda_e)/d\theta_o$ and $D_{12} = D_{21} = 0$.³ The angular diameter distance then simplifies into

$$\tilde{D}_A = \left| \frac{d\xi_1}{db} \left(\frac{d\theta_o}{db} \right)^{-1} \right|. \quad (\text{C.53})$$

To recover the angle $d\theta$ subtended by the two light rays separated by dl_s at the source, we can use Eq. (B.14) by considering light rays at the observer location, $\omega^\mu(\lambda_o, b)$ and $\omega^\mu(\lambda_o, b + \delta b)$, expanding up to second order in δb

$$1 - \cos \theta \simeq \frac{\delta\theta^2}{2} = -\frac{\omega^\mu(\lambda_o, b)\omega_\mu(\lambda_o, b + \delta b)}{u^\mu\omega_\mu(\lambda_o, b)u^\nu\omega_\nu(\lambda_o, b + \delta b)} \simeq -\frac{\overbrace{\omega^\mu\dot{\xi}_\mu}^{=0}\delta b + \frac{\omega^\mu(\lambda_o, b)}{2}\frac{\partial^2\omega_\mu}{\partial b^2}\delta b^2}{\omega_o^2}, \quad (\text{C.54})$$

where we defined $\omega_o = -g_{\mu\nu}u_o^\mu\omega^\nu(\lambda_o, b)$. Noticing

$$\omega_\mu \frac{\partial \dot{\xi}^\mu}{\partial b} = \frac{\partial}{\partial b} \left(\overbrace{\omega^\mu \dot{\xi}_\mu}^{=0} \right) - \dot{\xi}^\mu \dot{\xi}_\mu, \quad (\text{C.55})$$

we end up with

$$d\theta_o = \frac{db}{\omega_o} \sqrt{\dot{\xi}^\mu \dot{\xi}_\mu}, \quad (\text{C.56})$$

which agrees with the expected $\theta \sim \dot{\xi}/\omega_o$.

As application of the previous formalism, we now want to derive the angular diameter distances expressions for the Kottler metric discussed in App. B. Due to the singularity at $r = 0$, there will be some sort of arbitrariness for D_1 and D_{1s} . For a generic light ray, Eq. (B.8), and for a generic massive observer, Eq. (B.19), (we use the shorthand $J(r)$ specified in Eq. (B.20))

$$\dot{\xi}^\mu = \left(0, \mp \frac{Ab}{r^2\sqrt{1 - Ab^2/r^2}}, 0, \frac{1}{r^2}\right), \quad \omega_o = \frac{1}{A} \left(E \mp J \sqrt{1 - \frac{b^2}{r^2}A} - \frac{b_u b}{r^2} \right); \quad (\text{C.57})$$

with this,

$$d\theta = \frac{db}{\omega_o} \sqrt{\frac{Ab^2}{r^2(r^2 - Ab^2)} + \frac{1}{r^2}}. \quad (\text{C.58})$$

³In the D_e case, if $\varphi_o - \varphi_e \neq \pm\pi$, the motion on the θ direction cannot be recovered by simply changing the $\theta = \pi/2$ plane to a different plane, and one should explicitly consider the geodesic equation with varying θ . This will translate in possibly $D_{11} \neq D_{22}$, and $D_{12}, D_{21} \neq 0$. We either assume $\varphi_o - \varphi_e$ to be sufficiently close to $\pm\pi$ so that we can avoid this issue, or that, for this setting, we can simply define from the get-go the angular diameter distance as in Eq. (C.52).

We want to find s_1^μ , at first for a comoving source with $b_u = 0$. Set $s_1^\mu = (a_1, a_2, 0, a_3)$; the factors a_i can to be found via

$$\begin{aligned} s_1^\mu u_\mu = 0 &\implies a_1 = a_2 \frac{\sqrt{E^2 - A}}{AE}, \\ g_{\mu\nu} s_1^\mu s_1^\nu = 1 &\implies a_2 = E \sqrt{1 - r^2 a_3^2}, \\ s_1^\mu \omega_\mu = 0 &\implies a_3 = \frac{\sqrt{1 - r^2 a_3^2}}{bA} \left(\sqrt{E^2 - A} - E \sqrt{1 - \frac{b^2 A}{r^2}} \right). \end{aligned} \quad (\text{C.59})$$

Up to $\mathcal{O}(x^2)$,

$$\begin{aligned} a_1 &= \frac{Hb}{1 - Hr} \left(1 \pm \frac{v}{Hr} \pm v \right) + \mathcal{O}(x^3), \quad a_2 = \frac{b}{r} (1 + Hr) (1 \pm v \pm vHr) + \mathcal{O}(x^3), \\ a_3 &= \frac{1}{r} \left(1 - \frac{b^2 (1 + Hr)^2}{2r^2} \right) + \mathcal{O}(x^3). \end{aligned} \quad (\text{C.60})$$

With similar procedure, for a source with generic emitter, $b_u \neq 0$, $E \neq 1$ but for a light ray with $b = 0$ (relevant for $\tilde{D}_1$), we have (no x expansion)

$$s_1^\mu = \left(\frac{b_u}{(E - J)r}, \frac{Ab_u}{(E - J)r}, 0, \frac{1}{r} \right). \quad (\text{C.61})$$

For defining $\tilde{D}_1$, similar to what we did for the lens redshift in Eq. (B.43), we can use a light ray, starting at a radius $r_{\text{lens}} > r_{\text{Sc}}$, $\varphi = 0$, with $b = 0$ (straight light ray). The observer will be a near-comoving one with $E = E_o$, $b_u = 0$. Then

$$d\theta = (E_o + \sqrt{E_o^2 - A(r_o)}) \frac{db}{r_o}. \quad (\text{C.62})$$

We have to write an expression for the bundle $\varphi(\lambda, b)$, we have to ensure it ends at x_o ; this can be done via

$$\varphi(\lambda, b) - \varphi_o = \int_{r_o}^{r(\lambda, b)} \frac{dr'}{r' \sqrt{\frac{r'^2}{b^2} - A(r')}}. \quad (\text{C.63})$$

We can choose the parametrization of the ray bundle at λ_{lens} such that $r(\lambda_{\text{lens}}, b) = r_{\text{lens}}$ for all b sufficiently close to $b = 0$. With this choice, $\xi^r(\lambda_{\text{lens}}, b = 0) = 0$. Then, we can write

$$\left. \frac{d\varphi(r(\lambda_{\text{lens}}, b) = r_{\text{lens}})}{db} \right|_{b=0} = - \int_{r_{\text{lens}}}^{r_o} \frac{dr}{r^2} = - \left(\frac{1}{r_{\text{lens}}} - \frac{1}{r_o} \right); \quad (\text{C.64})$$

analogously, via Eq. (B.21), we can see⁴

$$\begin{aligned} t(\lambda, b) - t_o &= \int_{r(\lambda, b)}^{r_o} \frac{dr}{A(r) \sqrt{1 - b^2 A(r)/r^2}} \implies \frac{dt(\lambda_{\text{lens}}, b = 0)}{db} \\ &= \int_{r_{\text{lens}}}^{r_o} \frac{dr A(r) b / r^2}{A(r) (1 - b^2 A(r)/r^2)^{3/2}} \Big|_{b=0} - \frac{1}{A(r_{\text{lens}})} \underbrace{\frac{dr(\lambda_{\text{lens}}, b = 0)}{db}}_{=0} = 0; \end{aligned} \quad (\text{C.65})$$

⁴a $b \neq 0$ would make the travel path curved and hence longer, making $b = 0$ a point of minimum for the travel time t . Hence, here only $d\varphi/db$ contributes. The negative sign of $d\varphi/db$ can be understood as follows: a more negative b increases the starting angle of the geodesic (since bigger $|b|$ means a greater excursion of φ , with negative sign meaning the direction is clockwise), whereas a more positive b means a smaller angle, once you realize that positive b goes counter-clockwise.

we then have, using the vector in Eq. (C.61) for expressing ξ_1

$$dl_s = \xi_1(r(\lambda_s) = r_{\text{lens}}, b = 0) db = r_{\text{lens}} \frac{d\varphi(r(\lambda) = r_{\text{lens}}, b)}{db} \Big|_{b=0} db = - \left(1 - \frac{r_{\text{lens}}}{r_o}\right) db ; \quad (\text{C.66})$$

taking for example $r_{\text{lens}} \simeq r_c$, we have

$$\tilde{D}_1 = \frac{r_o}{E_o + \sqrt{E_o^2 - A(r_o)}} \left(1 - \frac{r_{\text{lens}}}{r_o}\right). \quad (\text{C.67})$$

Taking $r_{\text{lens}} \simeq r_c$, we have

$$\tilde{D}_1 = \frac{r_o}{1 + z_l} + \mathcal{O}(x). \quad (\text{C.68})$$

For $\tilde{D}_{\text{ls}}$, we can again take a light ray from the source with $b = 0$ towards the generic not comoving observer at r_{lens} . We have, using Eq. (C.60) with $b = 0$

$$d\theta = \frac{A(r_{\text{lens}})}{E + J(r_{\text{lens}})} \frac{db}{r_{\text{lens}}}, \quad (\text{C.69})$$

$$dl_s = \xi_1(r(\lambda_s) = r_s, b = 0) db = r_s \frac{d\varphi}{db} \Big|_{b=0} db = -r_s db \int_{r_{\text{lens}}}^{r_s} \frac{dr}{r^2} = \left(\frac{r_s}{r_{\text{lens}}} - 1\right) db ; \quad (\text{C.70})$$

and

$$\tilde{D}_{\text{ls}} = r_s \frac{(E + J(r_{\text{lens}}))}{A(r_{\text{lens}})} \left(1 - \frac{r_{\text{lens}}}{r_s}\right). \quad (\text{C.71})$$

Unfortunately, one sees that we cannot really get rid of the spurious distance r_{lens} by setting it to zero due to the $r = 0$ singularity; taking again $r_{\text{Sc}}/r_{\text{lens}} \sim \mathcal{O}(x)$ and $J = 0, E = 1 + \mathcal{O}(x)$ we have

$$\tilde{D}_{\text{ls}} = r_s(1 + \mathcal{O}(x)). \quad (\text{C.72})$$

Even though there are difficulties in giving some sense of $\tilde{D}_{\text{ls}}$, we remark that anyway $\tilde{D}_{\text{ls}}$ is unobservable, and what we need is just the unlensed result $D_{\text{ls}} = r_s$.

Finally, for D_s , we cannot set $b = 0$, but we can consider to write the result up to $\mathcal{O}(x^2)$; then, with Eq. (C.60),

$$\xi_1(\lambda_s) = -A(r_s) s_1^0 \frac{\partial t}{\partial b} + \frac{s_1^r}{A(r_s)} \frac{\partial r}{\partial b} + r_s^2 s_1^\varphi \frac{\partial \varphi}{\partial b} ; \quad (\text{C.73})$$

we can again take the affine parameter λ_s at the event of light ray emission to satisfy $\xi^r(\lambda_s, b) = 0$ around the actual b of the observed light ray; using Eq. (B.56), substituting $r_s \rightarrow r(\lambda, b)$, we see that the $\partial t/\partial b = s \partial t/\partial r_c + \mathcal{O}(x^2)$ term yields only $\mathcal{O}(x)$ terms,

$$\frac{\partial t(\lambda_s, b \simeq r_c)}{\partial b} = -s \left(2 \frac{r_{\text{Sc}}}{r_c} + r_c \frac{r_s + r_o}{r_s r_o} \right) + \mathcal{O}(x^2) ; \quad (\text{C.74})$$

and Eq. (B.32) for $\partial \varphi/\partial b$ (with $\varphi_o = 0$) we obtain

$$\frac{\partial \varphi}{\partial b} = \frac{r_s + r_o}{r_s r_o} + 2 \frac{r_{\text{Sc}}}{\tilde{b}^2} + \frac{15}{8} \pi \frac{r_{\text{Sc}}^2}{\tilde{b}^3} - \frac{r_{\text{Sc}}^3}{\tilde{b}^4} + \frac{\tilde{b}^2(r_o^3 + r_s^3)}{2r_s^3 r_o^3} + r_s \mathcal{O}(x^3), \quad (\text{C.75})$$

which, using

$$\frac{db}{d\theta} = \frac{r_o}{1 + z_l} + \frac{b^2(2Hr_o - 1)}{2r_o}, \quad (\text{C.76})$$

implies the angular diameter distance expression

$$\begin{aligned} \tilde{D}_s = & \frac{r_o}{1+z_l} \left(\left(\frac{r_s + r_o}{r_o} + 2 \frac{r_s r_{Sc}}{\tilde{b}^2} \right) \left(1 - \frac{\tilde{b}^2 (1 - 2Hr_o)(1+z_l)}{2r_o^2} \right) + \frac{15}{8} \pi \frac{r_s r_{Sc}^2}{\tilde{b}^3} - \frac{r_{Sc}^3 r_s}{\tilde{b}^4} \right. \\ & \left. + \frac{\tilde{b}^2 (r_o^3 + r_s^3)}{2r_o^3 r_s^2} + sH(1 + Hr_s) \left(2r_{Sc} + \frac{(r_s + r_o)\tilde{b}^2}{r_s r_o} \right) \right) + r_s \mathcal{O}(x^3). \end{aligned} \quad (C.77)$$

We see that the result is pretty involved. It seems like there is no real advantage in expressing the lens equation via these lensed angular diameter distances, especially if such distances are not deemed to be observable. A more thorough accounting of lensed angular diameter distances in Kottler metric is beyond the scope of this thesis.

Ref. [But16] claims that one needs to divide by 2 our result for $\tilde{D}_s$ to obtain the correct limit for de-Sitter. Our results at $\mathcal{O}(x)$ is consistent with theirs if $\beta = 0$, upon substitution of $\tilde{b}$ with θ_E of Eq. (B.40). However, we do not understand why one should sum over the observed angle of both images, which is how the factor 2 comes about. We just finally comment with the fact that, with $r_{Sc} = 0$ and in the $b \rightarrow 0$ limit for D_s , Eqs (C.67),(C.71),(C.77) reduce to the de-Sitter expressions (or unlensed angular diameter distances)

$$D_l = \frac{r_o}{1 + Hr_o}, \quad D_{ls} = r_s, \quad D_s = \frac{r_s + r_o}{1 + Hr_o}. \quad (C.78)$$

C.5 Light propagation in a curved background with plasma and axions

In this appendix, we review the propagation of light in a curved background, in the presence of plasma and axion particles which couple to electromagnetism. These notes were developed for investigating the plasma effects on the calculations of Refs. [MV20b; MV20a; FGR19] and possible applicability for eclipsing neutron star systems [Bur+03; Lyn+04; Bre+12; LT05; Low+24; Bai+20] for axion models like the one in Ref. [HH18], as part of an ongoing project with C. Sun.

C.5.1 The axion Maxwell equations

At the classical level, the propagation of light in an axion background follows from the action

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \partial_\mu a \partial^\mu a + \frac{c_{a\gamma}}{4} a F_{\mu\nu} \tilde{F}^{\mu\nu} - A_\mu j^\mu \right) =: \int d^4x \sqrt{-g} \mathcal{L}, \quad (C.79)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor, A_μ is the electromagnetic field, j^μ is an external current (from which plasma effects will arise), a is the axion field, $c_{a\gamma}$ is the axion-photon coupling and

$$\tilde{F}^{\mu\nu} = \frac{\epsilon^{\mu\nu\alpha\beta}}{2} F_{\alpha\beta}. \quad (C.80)$$

We are using the Levi Civita tensor notation

$$\epsilon_{\mu\nu\rho\sigma} = \sqrt{-g} \epsilon_{\mu\nu\rho\sigma}, \quad \epsilon_{0123} = 1, \quad (C.81)$$

where $\epsilon_{\mu\nu\rho\sigma}$ is a complete antisymmetric quantity. $\epsilon_{\mu\nu\rho\sigma}$ is not a tensor, and we will write it always with low indices. An useful relations involving $\epsilon_{\mu\nu\rho\sigma}$ is

$$\epsilon_{i_1 \dots i_p j_1 \dots j_q} \epsilon^{i_1 \dots i_p k_1 \dots k_q} = p! \begin{vmatrix} \delta_{j_1}^{k_1} & \dots & \delta_{j_q}^{k_1} \\ \vdots & \ddots & \vdots \\ \delta_{j_1}^{k_q} & \dots & \delta_{j_q}^{k_q} \end{vmatrix}; \quad (C.82)$$

moreover, notice that we can write

$$g^{\mu a} g^{\nu b} g^{\rho c} g^{\sigma d} \epsilon_{\mu\nu\rho\sigma} = \det(g^{-1}) \epsilon_{abcd} , \quad (\text{C.83})$$

with which

$$\epsilon^{\mu\nu\rho\sigma} = g^{\mu a} g^{\nu b} g^{\rho c} g^{\sigma d} \epsilon_{abcd} = -\frac{1}{\sqrt{-g}} \epsilon_{\mu\nu\rho\sigma} . \quad (\text{C.84})$$

The metric we assume will be the Newtonian

$$ds^2 = -dt^2 (1 - 2\Phi) + (1 + 2\Phi) dx^i dx^i ; \quad (\text{C.85})$$

we will thus consider systems where cosmological effects can be neglected. We do this just for simplicity.

From the equation of motions for the electromagnetic field A_μ

$$\frac{\partial \sqrt{-g} \mathcal{L}}{\partial A_\mu} = \partial_\nu \frac{\partial \sqrt{-g} \mathcal{L}}{\partial \partial_\nu A_\mu} , \quad (\text{C.86})$$

we obtain⁵

$$-\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}) - \frac{c_{a\gamma}}{\sqrt{-g}} \epsilon_{\nu\mu\alpha\beta} \partial_\nu a \partial_\alpha A_\beta = j^\mu , \quad (\text{C.88})$$

which can be rewritten as, using Eq. (C.84)

$$-\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta}) + c_{a\gamma} \epsilon^{\nu\mu\alpha\beta} \partial_\nu a \frac{F_{\alpha\beta}}{2} = j^\mu . \quad (\text{C.89})$$

The charge density an observer with 4-velocity u^μ sees is

$$\rho = -u_\mu j^\mu ; \quad (\text{C.90})$$

j^μ , as we already mentioned, encodes plasma effects. If we choose a 4-velocity u^μ for which the plasma is at rest, we can split j^μ as

$$j^\mu = \rho u^\mu + J^\mu , \quad J^\mu u_\mu = 0 , \quad J^\mu = (g^\mu_\nu + u^\mu u_\nu) j^\nu . \quad (\text{C.91})$$

With such a u^μ , we can write the electric and magnetic field as

$$E_\mu = -F_{\mu\nu} u^\nu , \quad B_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} u^\nu F^{\rho\sigma} = \tilde{F}_{\mu\nu} u^\nu . \quad (\text{C.92})$$

Notice that, as defined, both E_μ and B_μ are purely spatial,

$$E_\mu u^\mu = 0 , \quad B_\mu u^\mu = 0 , \quad (\text{C.93})$$

and that we can express

$$F_{\mu\nu} = u_\nu E_\mu - u_\mu E_\nu - \epsilon_{\mu\nu\rho\sigma} u^\rho B^\sigma . \quad (\text{C.94})$$

Also,

$$\tilde{F}_{\mu\nu} = u_\mu B_\nu - u_\nu B_\mu - \epsilon_{\mu\nu\rho\sigma} u^\rho E^\sigma . \quad (\text{C.95})$$

⁵To derive it, one can use the relation

$$\epsilon_{\gamma\delta\alpha\beta} F_{\alpha\beta} F_{\gamma\delta} = 4 \epsilon_{\gamma\delta\alpha\beta} \partial_\alpha A_\beta \partial_\gamma A_\delta , \quad \sqrt{-g} \epsilon^{\gamma\delta\alpha\beta} \partial_\nu (a \frac{\partial}{\partial \partial_\nu A_\mu} (\partial_\alpha A_\beta \partial_\gamma A_\delta)) = -2 \epsilon_{\nu\mu\alpha\beta} \partial_\nu a \partial_\alpha A_\beta . \quad (\text{C.87})$$

We can express $\vec{E}$ and $\vec{B}$ using the local rest reference frame of the observer inside the plasma, or the tetrad $e^{\hat{a}} = e^{\hat{a}}_{\mu} dx^{\mu}$. Notice $e^{\mu}_{\hat{0}} = u^{\mu}$, $e^{\hat{0}}_{\mu} e^{\mu}_{\hat{0}} = 1 = u^{\mu} e^{\hat{0}}_{\mu} \implies e^{\hat{0}}_{\mu} = -u_{\mu}$, and $g_{\mu\nu} = \eta_{\hat{a}\hat{b}} e^{\hat{a}}_{\mu} e^{\hat{b}}_{\nu}$. Then,

$$E^{\hat{i}} = e^{\hat{i}}_{\nu} E^{\nu} = F^{\mu\nu} u_{\mu} e^{\hat{i}}_{\nu} = -F^{\mu\nu} e^{\hat{0}}_{\mu} e^{\hat{i}}_{\nu} = -F^{\hat{0}\hat{i}}, \quad (\text{C.98})$$

$$B^{\hat{i}} = e^{\hat{i}}_{\mu} B^{\mu} = \frac{1}{2} \sqrt{-g} \in_{\alpha\nu\rho\sigma} e^{\nu}_{\hat{0}} e^{\rho}_{\hat{j}} e^{\sigma}_{\hat{k}} F^{\hat{j}\hat{k}} e^{\hat{i}}_{\mu} \underbrace{g^{\alpha\mu}}_{=\eta^{\hat{a}\hat{b}} e^{\hat{a}}_{\alpha} e^{\hat{b}}_{\mu}} = -\frac{1}{2} \in_{ijk} F^{\hat{j}\hat{k}}, \quad (\text{C.99})$$

where hatted indices denote tetrad indices, and we used Eq. (C.96) for the magnetic field. This, unsurprisingly, is the same result one obtains in Minkowski (for reference, $F_{\hat{j}\hat{k}} = -\in_{jki} B^{\hat{i}}$). Also,

$$j^0 = \frac{\rho}{\sqrt{-g_{00}}} = e^0_{\hat{0}} \rho, \quad j^i = e^i_{\hat{j}} J^{\hat{j}}. \quad (\text{C.100})$$

With this formalism, we can write the relation for j^{μ} for a plasma as

$$\partial_{\hat{0}} \rho = -\partial_{\hat{i}} J^{\hat{i}}, \quad J^{\hat{i}} = \sigma E^{\hat{i}}, \quad \sigma = i\omega_p^2 / \omega, \quad (\text{C.101})$$

where we defined $\partial_{\hat{\mu}} := e^{\nu}_{\hat{\mu}} \partial_{\nu}$. We remark that the previous holds in the local rest frame of the plasma.

Specializing the tetrad to the metric Eq. (C.85), always keeping first order in the gravitational potential Φ , we have

$$e^{\hat{0}}_{\mu} = (1 + \Phi) \delta_{0\mu} = -u_0 \delta_{0\mu}, \quad e^{\hat{i}}_j = (1 - \Phi) \delta_{ij}. \quad (\text{C.102})$$

The equations we are about to derive describe the $\vec{E}$ and $\vec{B}$ propagation on the plasma as seen by observers following, point by point, the propagating $F_{\mu\nu}$. With such a choice,

$$F_{0i} = \sqrt{-g_{00}} g_{ij} e^j_{\hat{k}} E^{\hat{k}} = E^{\hat{i}}, \quad F_{ij} = -\frac{\sqrt{-g}}{\sqrt{-g_{00}}} \in_{ijk} e^k_{\hat{l}} B^{\hat{l}} = -(1 - 2\Phi) \in_{ijk} B^{\hat{k}}. \quad (\text{C.103})$$

We can recast Eq. (C.89) in the Maxwell form, up to first order in Φ and neglecting Φ derivatives, to obtain the axion Maxwell equations in curved background

$$\nabla \cdot \vec{E} = (1 - \Phi) \rho - c_{a\gamma} \nabla a \cdot \vec{B}, \quad (\text{C.104})$$

$$-\frac{d\vec{E}}{dt} + (1 + 2\Phi) \nabla \times \vec{B} = (1 + \Phi) \vec{J} + c_{a\gamma} \dot{a} \vec{B} + c_{a\gamma} (1 + 2\Phi) \nabla a \times \vec{E}, \quad (\text{C.105})$$

$$(1 - 2\Phi) \nabla \cdot \vec{B} = 0, \quad (\text{C.106})$$

$$(1 - 2\Phi) \frac{d\vec{B}}{dt} + \nabla \times \vec{E} = 0. \quad (\text{C.107})$$

The last two come from the constraint equation (considering $\mu, \nu, \rho \neq 0$ and $\mu = 0, \nu, \rho \neq 0$ respectively)

$$\partial_{\mu} F_{\nu\rho} + \partial_{\nu} F_{\rho\mu} + \partial_{\rho} F_{\mu\nu} = 0. \quad (\text{C.108})$$

⁶With the tetrad, the determinant of the metric obeys

$$\in_{\alpha\beta\gamma\delta} \det g^{-1} = \in_{\mu\nu\rho\sigma} \eta^{ab} e^{\mu}_{\hat{a}} e^{\nu}_{\hat{b}} e^{\rho}_{\hat{c}} e^{\sigma}_{\hat{d}} \eta^{cd} e^{\hat{a}}_{\alpha} e^{\hat{b}}_{\beta} e^{\hat{c}}_{\gamma} e^{\hat{d}}_{\delta} \eta^{ef} e^{\hat{e}}_{\alpha} e^{\hat{f}}_{\beta} e^{\hat{g}}_{\gamma} e^{\hat{h}}_{\delta} = \det\{e^{\mu}_{\hat{a}}\} \in_{aceg} \eta^{ab} \eta^{cd} \eta^{ef} \eta^{gh} e^{\hat{a}}_{\alpha} e^{\hat{b}}_{\beta} e^{\hat{c}}_{\gamma} e^{\hat{d}}_{\delta} = -(\det\{e^{\mu}_{\hat{a}}\})^2 \in_{\alpha\beta\gamma\delta}, \quad (\text{C.96})$$

which implies

$$\det g^{-1} = -(\det\{e^{\mu}_{\hat{a}}\})^2. \quad (\text{C.97})$$

C.5.2 Refractive index

We now apply the geometric optics limit to the field $F_{\mu\nu}$, via the ansatz

$$F_{\mu\nu} = \text{Re}(e^{iS/\epsilon} f_{\mu\nu}), \quad f_{\mu\nu} = \sum_{n=0}^{\infty} \epsilon^n f_{\mu\nu}^{(n)}, \quad (\text{C.109})$$

where ϵ is a bookkeeping expansion parameter, S is a scalar function and $f_{\mu\nu}^{(n)}$ antisymmetric tensors. One defines the light ray wavevector as

$$k_\mu = \frac{1}{\epsilon} \partial_\mu S; \quad (\text{C.110})$$

defined like this, k_μ carries a $1/\epsilon$ factor, explicitly showing that it is a dominant term in the geometric optics limit. The frequency of a light ray as seen by an observer u^μ reads

$$\omega = -k_\mu u^\mu; \quad (\text{C.111})$$

with u^μ as the 4-vector associated to the local rest frame of the plasma, we have

$$\omega = \sqrt{-g_{00}} k^0 = e_0^0 k^0 = -k^{\hat{0}}, \quad (\text{C.112})$$

and the refractive index reads [BH75]

$$n^2 = 1 + \frac{k^\mu k_\mu}{(k_\mu u^\mu)^2} = \frac{k_\perp^2}{\omega^2}, \quad k^\mu =: \omega u^\mu + k_\perp^\mu; \quad (\text{C.113})$$

notice that

$$k_\perp^2 = g_{ij} k_\perp^i k_\perp^j = g_{ij} k^i k^j = k^{\hat{i}} k^{\hat{i}} =: \hat{k}^2 \implies n = \frac{\hat{k}}{\omega}. \quad (\text{C.114})$$

From Eq. (C.109), together with Eqs. (C.98) and (C.104) for ρ , we can write

$$\vec{E} = \text{Re}(e^{iS/\epsilon} \sum_{n=0}^{\infty} \epsilon^n \vec{E}^{(n)}), \quad \vec{B} = \text{Re}(e^{iS/\epsilon} \sum_{n=0}^{\infty} \epsilon^n \vec{B}^{(n)}), \quad \rho = \text{Re}(e^{iS/\epsilon} \sum_{n=0}^{\infty} \epsilon^n \rho^{(n)}). \quad (\text{C.115})$$

The plasma effect comes from the free electrons reacting to the incoming electromagnetic wave, hence we expect to be able to write the same expansion for ρ we can write for $F_{\mu\nu}$. Notice that, in the lowest order in ϵ ,

$$\partial_{\hat{0}} \rho = ie_0^0 k_0 \rho = -i\omega \rho. \quad (\text{C.116})$$

The dispersion relation is recovered from the Maxwell equations at order $1/\epsilon$. Assuming a derivatives and σ to be order $1/\epsilon$, we can rewrite Maxwell equations Eqs. (C.104), (C.105), (C.106), (C.107) as

$$ik^{\hat{i}} E^{\hat{i}} \left(1 - \frac{\omega_p^2}{\omega^2}\right) = -c_{a\gamma} B^{\hat{i}} \partial_{\hat{i}} a, \quad (\text{C.117})$$

$$i\omega E^{\hat{i}} \left(1 - \frac{\omega_p^2}{\omega^2}\right) = -i \epsilon_{ijk} k^{\hat{j}} B^{\hat{k}} + c_{a\gamma} B^{\hat{i}} \partial_{\hat{0}} a + c_{a\gamma} \epsilon_{ijk} E^{\hat{k}} \partial_{\hat{j}} a, \quad (\text{C.118})$$

$$k^{\hat{i}} B^{\hat{i}} = 0, \quad (\text{C.119})$$

$$-\omega B^{\hat{i}} + \epsilon_{ijk} k^{\hat{j}} E^{\hat{k}} = 0, \quad (\text{C.120})$$

where we exploited (we now set $\epsilon = 1$)

$$k_0 = \partial_0 S = -(1 + \Phi)\omega, \quad k_i = \partial_i S = (1 - \Phi)k^{\hat{i}}, \quad \partial_{\hat{0}} = (1 - \Phi)\partial_0, \quad \partial_{\hat{i}} = (1 + \Phi)\partial_i, \quad (\text{C.121})$$

which imply

$$\partial_{\hat{0}} E^{\hat{i}} = -i\omega E^{\hat{i}}, \quad \partial_{\hat{i}} E^{\hat{j}} = ik^{\hat{i}} E^{\hat{j}}, \quad (\text{C.122})$$

and similar for $B^{\hat{i}}$. Moreover, we used, from Eq. (C.101)

$$\partial_{\hat{0}} \rho = -i\omega \rho = -\partial_{\hat{i}} J^{\hat{i}} \implies \rho = i \frac{\omega_p^2}{\omega^2} k^{\hat{i}} E^{\hat{i}}. \quad (\text{C.123})$$

Notice that, as expected, the gravitational potential disappears when everything is expressed with respect to tetrad quantities.

Now derive with respect to $\partial_{\hat{0}}$ Eq. (C.120) and use all the other Maxwell equations Eq. (C.117)-(C.119) to obtain, at order $1/\epsilon^2$, (we are neglecting derivatives of k_{μ})

$$\begin{aligned} (\omega^2 - \omega_p^2 - \hat{k}^2) B^{\hat{i}} - ic_{a\gamma} \left(k^{\hat{j}} \partial_{\hat{j}} a + \left(1 - \frac{\omega_p^2}{\omega^2} \right) \omega \partial_{\hat{0}} a \right) E^{\hat{i}} + c_{a\gamma}^2 \left((\partial_{\hat{0}} a)^2 B^{\hat{i}} + \epsilon_{ijk} \partial_{\hat{0}} a \partial_{\hat{j}} a E^{\hat{k}} \right. \\ \left. - \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1} \partial_{\hat{i}} a \partial_{\hat{j}} a B^{\hat{j}} \right) = 0; \end{aligned} \quad (\text{C.124})$$

analogously, deriving with respect to $\partial_{\hat{0}}$ Eq. (C.118) and using the other ones, we obtain

$$\begin{aligned} (\omega^2 - \omega_p^2 - \hat{k}^2) E^{\hat{i}} + c_{a\gamma} (i\omega \partial_{\hat{0}} a + i \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1} k^{\hat{j}} \partial_{\hat{j}} a + (\partial_{\hat{0}} a)^2) B^{\hat{i}} \\ - \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1} c_{a\gamma}^2 (\partial_{\hat{j}} a \partial_{\hat{j}} a) E^{\hat{i}} + c_{a\gamma}^2 \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1} \partial_{\hat{i}} a \partial_{\hat{j}} a E^{\hat{j}} + c_{a\gamma} \epsilon_{ijk} (\partial_{\hat{j}} \partial_{\hat{0}} a) E^{\hat{k}} = 0. \end{aligned} \quad (\text{C.125})$$

Ignoring second order terms on $c_{a\gamma}$ and double derivatives of the axion, we recover the system of equations

$$\begin{pmatrix} \omega^2 - \hat{k}^2 - \omega_p^2 & ic_{a\gamma} (\partial_{\hat{0}} a \omega + \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1} k^{\hat{i}} \partial_{\hat{i}} a) \\ -ic_{a\gamma} \left(\left(1 - \frac{\omega_p^2}{\omega^2} \right) \partial_{\hat{0}} a \omega + k^{\hat{i}} \partial_{\hat{i}} a \right) & \omega^2 - \hat{k}^2 - \omega_p^2 \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (\text{C.126})$$

Solving for $\hat{k}$, defining $k^{\hat{i}} =: \hat{k} v^i$, with v^i the unit vector in the direction of $k^{\hat{i}}$, we find

$$\hat{k} = \omega \left(\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \pm \frac{c_{a\gamma}}{2\omega} \left| (\partial_{\hat{0}} a + \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1/2} v^i \partial_{\hat{i}} a) \right| \right) + \mathcal{O}(c_{a\gamma}^2, \dots). \quad (\text{C.127})$$

This implies the refractive index

$$n = \frac{\hat{k}}{\omega} = \left(\sqrt{1 - \frac{\omega_p^2}{\omega^2}} \pm \frac{c_{a\gamma}}{2\omega} \left| \partial_{\hat{0}} a + \left(1 - \frac{\omega_p^2}{\omega^2} \right)^{-1/2} v^i \partial_{\hat{i}} a \right| \right) + \mathcal{O}(c_{a\gamma}^2, \dots). \quad (\text{C.128})$$

For reference, we can write ω as a function of $\hat{k}$

$$\omega = \sqrt{\hat{k}^2 + \omega_p^2} \mp \frac{c_{a\gamma}}{2} \left| \left(1 + \frac{\omega_p^2}{\hat{k}^2} \right)^{-1/2} \partial_0 a + v^i \partial_i a \right| + \mathcal{O}(c_{a\gamma}^2, \dots). \quad (\text{C.129})$$

The term which may be interesting to study are the plasma-axion mixing term $\propto \omega_p^2 c_{a\gamma}$; it is interesting to investigate possible novel phenomenological signals coming from such terms.

C.5.3 The Hamilton equations for light propagation

From the definition of refractive index Eq. (C.113), one can build an Hamiltonian $\mathcal{H}(x^\mu, k_\nu)$ [BH75; Syn60; BPB22]

$$\mathcal{H}(x^\rho, k_\sigma) = \frac{1}{2} (g^{\mu\nu} k_\mu k_\nu - (n^2(\omega, x^\rho) - 1)(k_\mu U^\mu(x^\rho))^2) = 0, \quad (\text{C.130})$$

where the last equality comes from a rearrangement of Eq. (C.113). Eq. (C.130) is a non-linear partial differential equation for the phase S (recall $k_\mu = \partial_\mu S$), and as such can be solved via the method of characteristic curves. For the case at hand, the characteristic curves are just Hamilton equations

$$\frac{dx^\mu}{d\lambda} = \frac{\partial \mathcal{H}}{\partial k_\mu}, \quad \frac{dk_\mu}{d\lambda} = -\frac{\partial \mathcal{H}}{\partial x^\mu}, \quad (\text{C.131})$$

where k^μ and x^μ are considered as independent variables. We immediately see that, differently from the non-plasma case, we have

$$\frac{dx^\mu}{d\lambda} = k^\mu + U^\mu n \frac{\partial n}{\partial \omega} (k_\rho U^\rho)^2 - U^\mu (n^2 - 1)(k_\rho U^\rho). \quad (\text{C.132})$$

(The previous holds only if n depends explicitly only on ω and not on the full k^μ).

As an application of such formalism, we now set the stage for computing the time delay between two pulses, one passing through an axion cloud and the other not; this can be a target for possible signals coming from the axion-plasma mixing term.

We can define the 4-vectors u^μ corresponding to the plasma-axion rest frame and v^μ the 4-vector corresponding to the observer. Assuming no peculiar motion of the plasma, we can write

$$v^\mu = (1, 0); \quad u^\mu = \left(\frac{1}{\sqrt{-g_{00}}}, 0 \right) \simeq (1 - \Phi, 0), \quad (\text{C.133})$$

where Φ influence is only at the plasma-axion location, and we expanded up to first order in Φ . In such a simple metric, the proper time $\Delta\tau'$ the observer experiences between observation of pulse 1 and 2 is trivially

$$\Delta\tau' = \int_{t'_1}^{t'_2} d\lambda = \int_{t'_1}^{t'_2} \frac{dt}{v^0} = t'_2 - t'_1. \quad (\text{C.134})$$

Instead, the proper time $\Delta\tau$ between emission of the two pulses at the emitter location is

$$\Delta\tau = \int_{t_1}^{t_2} d\lambda = \int_{t_1}^{t_2} \frac{dt}{u^0} = (1 + \Phi(x_E))(t_2 - t_1), \quad (\text{C.135})$$

where we recognize the gravitational redshift factor (and we assumed Φ to be time-independent). We can then write

$$\Delta\tau' = \Delta t_2 - \Delta t_1 - \frac{\Delta\tau}{1 + \Phi(x_E)}, \quad \Delta t_i := t'_i - t_i. \quad (\text{C.136})$$

We assume we model $\Delta\tau$, and that we know x_e (distance of the axion cloud with respect to us) and we are free to set $x_o^\mu = 0$, which is the coordinates (time and space) of the observer. Then, the light ray equations come from Eq. (C.131), which we can write as (we neglect derivatives of Φ , and we suppress the subscript i)

$$\begin{aligned}\frac{dx^\mu}{d\lambda} &= k^\mu - \omega \frac{\partial \omega}{\partial k_\mu} - n \frac{\partial n}{\partial k_\mu} \omega^2 ; \\ \frac{dk_\mu}{d\lambda} &= \omega \frac{\partial \omega}{\partial x^\mu} + n \frac{\partial n}{\partial x^\mu} \omega^2 ;\end{aligned}\tag{C.137}$$

n is given by Eq. (C.128), and given the fluctuating axion field, it is a function of $t(\lambda)$ and $x(\lambda)$. $\omega = -k^\mu u_\mu(x^\nu) \simeq k^0(1 + \Phi(x^\nu)) =: \omega(x^\nu)$, where we remark that the coordinate dependence of ω comes solely on u^μ , as in this formalism k_μ and x^μ are considered independent degrees of freedom. Parametrizing $x_i^\mu(\lambda = 0) = (t_i, x_{ei}^j)$, $(t'_i, x_o^j = 0) =: x_i^\mu(\lambda_i^o)$, supposing we have a monochromatic pulse with $\omega(\lambda = \lambda_i^o) = \omega_o$, we need to solve the system of equations Eq. (C.137) with the final conditions

$$x_i^j(\lambda_i^o) = x_o^j, \omega_i(\lambda_i^o) = \omega_o, \frac{1}{2}(g^{\mu\nu}k_\mu k_\nu - (n^2(\omega, x^o) - 1)(k_\mu U^\mu(x^o))^2) \Big|_{\lambda=\lambda_i^o} = 0 ;\tag{C.138}$$

in a generic setting, the light ray can arrive at a certain angle in the sky with respect to the deflecting object line of sight due to lensing effects. This allows us to connect e.g. $k^2(\lambda_{oi})$, $k^3(\lambda_{oi})$ to such observed angles. For the time, without losing generality, we can take $t_1(\lambda = \lambda_1^o) = 0$, so that $t_2(\lambda = \lambda_2^o) = \Delta t$. We can then evolve the system of equation until λ_i^o , the latter to be found such that the corresponding $x_1^1(0) = x_e$ if e.g. we suppose we know the distance to the source and x^1 is the coordinate connecting observer and lens. Finally, we can write

$$\Delta t_i = \int_{t_i}^{t'_i} dt = \int_0^{\lambda_i^o} \frac{dt}{d\lambda} d\lambda ,\tag{C.139}$$

and we have all the ingredients to compute the observable $\Delta\tau'$.

A simplified case. In order to give an estimate of the magnitude of the time travel time Eq. (C.139), suppose we have no gravity, n is independent on time and the geodesic motion happens only on the x direction. Then

$$\mathcal{H}(\omega, x, t) = \frac{1}{2}(k^2 - n^2(\omega, x, t)\omega^2) , k := k^1 , \omega = k^0 = -k_0 ;\tag{C.140}$$

the Hamilton equations read

$$\frac{dt}{d\lambda} = \omega^2 n \frac{\partial n}{\partial \omega} + \omega n^2 ;\tag{C.141}$$

$$\frac{dx}{d\lambda} = k = n\omega ;\tag{C.142}$$

$$\frac{d\omega}{d\lambda} = -\omega^2 n \frac{\partial n}{\partial t} ;\tag{C.143}$$

$$\frac{dk}{d\lambda} = \omega^2 n \frac{\partial n}{\partial x} ;\tag{C.144}$$

the last equation is actually redundant, since we have the constraint $k = n\omega$. By the implicit function theorem, if we define

$$F(x, \lambda) = x - x(\lambda) = 0 ,\tag{C.145}$$

then if $\frac{df}{d\lambda} = k$ is everywhere different from zero, we can find a function $\lambda(x)$ (in other words, there is a one to one correspondence between values of x and values of λ). Then, the derivative

$$\frac{dt}{dx} = \frac{dt(\lambda(x))}{dx} = \frac{dt}{d\lambda} \left(\frac{dx}{d\lambda} \right)^{-1}, \quad (\text{C.146})$$

is well-defined, and we can write

$$\begin{aligned} \Delta t &= \int_{x_e}^{x_o} \frac{dt}{dx} dx = \int_{x_e}^{x_o} \left(n + \omega \frac{\partial n}{\partial \omega} \right) dx \\ &\simeq \int_{x_e}^{x_o} dx \left(1 + \frac{\omega_p^2(x)}{2\omega^2(x, t(x))} \mp \frac{c_{a\gamma}}{2\omega(x, t(x))} (\partial_x a) \frac{\omega_p^2(x)}{\omega^2(x, t(x))} \right); \end{aligned} \quad (\text{C.147})$$

and the contribution due to the plasma-axion mixing term reads

$$\Delta t_{\text{mixed}} \simeq \mp \frac{c_{a\gamma}}{2} \int_{x_e}^{x_{e+}} dx \frac{\omega_p^2(x)}{\omega^3(x, t(x))} \partial_x a(x, t(x)), \quad (\text{C.148})$$

where $x_e - x_{e+}$ is the extension of the axion cloud. Notice that, without plasma, the axion contribution at this order would be zero. The previous equation only gives a rough estimate of the axion-plasma mixing term, as we neglected time-variability, $\vec{k}$ dependence of n etc., but it gives an idea on the time delay size one could expect from a particular situation.

Notice that one should impose that the light ray does never cross areas with $\omega < \omega_p$, where it would get reflected.

Appendix D

Gravitational Lens Profiles and the Complex Notation

D.1 Inserting ULDM profiles in lenstronomy

In Ch. 5, we performed mock analysis using the gravitational lensing software `lenstronomy` [BAR16; BA18a; Bir+21]. As a part of the work of Ref. [BT21b], density profiles pertaining the ULDM soliton were added to `lenstronomy`, which we describe in App. D.1.1, D.1.2; moreover, we implemented a different method to consider a MSD model in the Bayesian inference pipeline, which does not suffer from problems related to the degeneracies of an approximate MSD model, described in App. D.1.3.

D.1.1 Soliton cored profile

We describe here the lensing properties of the following density profile, which maps to the soliton profile in the presence of a PL background ansatz in Eq. (4.6)

$$\rho = \rho_0 \left(1 + f^2 \left(\frac{r}{r_c} \right)^2 \right)^{-\gamma}; \quad (\text{D.1})$$

f can be defined such that r_c is the half-light radius, which implies $f^2 = 0.5^{-1/\gamma} - 1$.

The convergence for this profiles reads ($r^2 = D_1 |\vec{\theta}|^2 + l^2$)

$$\begin{aligned} \Sigma_{\text{cr}} \kappa(\vec{\theta}) &= \rho_0 \int_{-\infty}^{\infty} dl \left(1 + f^2 \left(\frac{|D_1 \vec{\theta}|^2 + l^2}{r_c} \right)^2 \right)^{-\gamma} \\ &= \frac{\rho_0 r_c / f}{\left(1 + f^2 \left(\frac{|D_1 \vec{\theta}|^2 + l^2}{r_c} \right)^2 \right)^{\gamma-1/2}} \int_{-\infty}^{\infty} \frac{dl'}{(1 + l'^2)^\gamma}; \end{aligned} \quad (\text{D.2})$$

with the substitution for l' and the value of last integral given by

$$l' = \frac{fl}{r_c \sqrt{1 + f^2 \left(\frac{|D_1 \vec{\theta}|^2}{r_c} \right)}}, \quad \int_{-\infty}^{\infty} \frac{dl'}{(1 + l'^2)^\gamma} = \frac{\sqrt{\pi} \Gamma(\gamma + 1/2)}{\Gamma(\gamma)},$$

we end up with

$$\kappa = \frac{\rho_0 r_c / f}{\Sigma_{\text{cr}}} \left(1 + f^2 \left(\frac{D_1 \theta}{r_c} \right)^2 \right)^{-\gamma+1/2}. \quad (\text{D.3})$$

By writing schematically $\kappa = \kappa_0(1 + A^2\theta^2)^{-\gamma+1/2}$, the radial deflection angle is, using (2.37),

$$\begin{aligned}\alpha(\theta) &= \frac{2\kappa_0}{\theta} \int_0^\theta \frac{\theta' d\theta'}{(1 + A^2\theta'^2)^{\gamma-1/2}} = \frac{2\kappa_0}{A^2\theta} \left(-\frac{1}{2\gamma-3} (1 + \theta'^2)^{-\gamma+3/2} \right) \Big|_0^{A\theta} \\ &= \frac{2}{2\gamma-3} \frac{\kappa_0(\theta_c/f)^2}{\theta} \left(1 - \left(1 + f^2 \left(\frac{\theta}{\theta_c} \right)^2 \right)^{-\gamma+3/2} \right) \xrightarrow{\theta \ll \theta_c} \kappa_0\theta,\end{aligned}\quad (\text{D.4})$$

where we defined $\theta_c := D_1/r_c$. The lensing potential is

$$\Psi(\theta) = \int_0^\theta d\theta' \alpha(\theta') = \frac{2}{2\gamma-3} \kappa_0(\theta_c/f)^2 \int^{f\theta/\theta_c} \frac{dy}{y} \left(1 - (1 + y^2)^{-\gamma+3/2} \right) \xrightarrow{\theta \ll \theta_c} \frac{1}{2} \kappa_0\theta^2; \quad (\text{D.5})$$

it admits a hypergeometric function representation

$$\Psi(\theta) = \frac{1}{2} \kappa_0\theta^2 {}_2F_3 \left(\{1, 1, \gamma - 1/2\}, \{2, 2\}, -\left(\frac{f\theta}{\theta_c} \right)^2 \right), \quad (\text{D.6})$$

where

$${}_qF_p(\{a_1, \dots, a_q\}, \{b_1, \dots, b_p\}, x) := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}, \quad (a)_n := \Gamma(a+n)/\Gamma(a). \quad (\text{D.7})$$

Notice that we can write all lensing quantities with the angular variables θ_c and $\kappa_0 := \rho_0 r_c / (\Sigma_{\text{cr}} f)$.

The hessian for any spherical profile, and in particular for this profile, can be written (remember $\vec{\alpha} = \alpha(\theta)\vec{\theta}/\theta$)

$$\frac{\partial \alpha_i}{\partial \theta_j} = \frac{\partial}{\partial \theta_j} \left(\alpha(\theta) \frac{\theta_i}{\theta} \right) = \alpha(\theta) \left(\frac{\delta_{ij}}{\theta} - \frac{\theta_i \theta_j}{\theta^3} \right) + \frac{\theta_i}{\theta} \frac{\partial \theta}{\partial \theta_j} \frac{\partial \alpha}{\partial \theta} = \alpha(\theta) \left(\frac{\delta_{ij}}{\theta} - \frac{\theta_i \theta_j}{\theta^3} \right) + \frac{\theta_i \theta_j}{\theta^2} \frac{d\alpha}{d\theta}. \quad (\text{D.8})$$

Notice that indeed $\partial \alpha_i / \partial x_i = 2\kappa$, as it should.

The mass, instead, will be

$$M = 4\pi\rho_0 \left(\frac{r_c}{f} \right)^3 \int_0^{fr/r_c} \frac{dr' r'^2}{(1 + r'^2)^\gamma} = \frac{4}{3} \rho_0 \pi (D_1 \theta)^3 {}_2F_1(\{3/2, \gamma\}, \{5/2\}, f\theta/\theta_c). \quad (\text{D.9})$$

D.1.2 Cored exponential profile

Another possible cored model is represented by

$$\rho = \rho_0 e^{-2a\lambda^2 m^2 r^2}. \quad (\text{D.10})$$

The convergence for this profile reads

$$\kappa = \left(\frac{\pi}{2a} \right)^{1/2} \frac{\rho_0}{\lambda m \Sigma_{\text{cr}}} e^{-2a\lambda^2 m^2 D_1^2 \theta^2} =: \pi^{1/2} \frac{\rho_0}{(2a)^{1/2} \lambda m \Sigma_{\text{cr}}} e^{-(\theta/\theta_c)^2}; \quad (\text{D.11})$$

defining

$$\theta_c^{-1} = (2a)^{1/2} \lambda m D_1, \quad \rho_\theta := \rho_0 D_1 / \Sigma_{\text{cr}}, \quad (\text{D.12})$$

we can rewrite Eq. (D.11) as

$$\kappa = \pi^{1/2} \rho_\theta \theta_c e^{-(\theta/\theta_c)^2}; \quad (\text{D.13})$$

the deflection angle, from Eq. (2.37), is instead

$$\alpha = \kappa_0 \frac{\theta_c^2}{\theta} \left(1 - e^{-(\theta/\theta_c)^2} \right), \quad \kappa_0 = \sqrt{\pi} \rho_\theta \theta_c. \quad (\text{D.14})$$

The lensing potential reads

$$\Psi(\theta) = \kappa_0 \theta_c^2 \int^{\theta/\theta_c} \frac{dy}{y} (1 - e^{-y^2}) = \kappa_0 \theta_c^2 \left(\frac{1}{2} \Gamma \left(0, \left(\frac{\theta}{\theta_c} \right)^2 \right) + \ln \frac{\theta}{\theta_c} \right) + \text{const.}, \quad (\text{D.15})$$

where

$$\Gamma(a, x) := \int_x^\infty t^{a-1} e^{-t} dt, \quad (\text{D.16})$$

is the upper incomplete gamma function. Notice that $\Gamma(0, x) = E_1(x)$, the exponential integral. Also notice the expansion

$$E_1(x) = -\gamma - \ln x - \sum_{k=1}^{\infty} \frac{(-x)^k}{k!k} \implies \Psi(\theta) \xrightarrow{\theta \ll \theta_c} \frac{\kappa_0}{2} \theta^2 + \text{const.} \quad (\text{D.17})$$

The hessian for this exponential profile reads

$$\frac{\partial \alpha_x}{\partial y} = \frac{\partial \alpha_y}{\partial x} = f_1 xy - 2f_2 xy, \quad \frac{\partial \alpha_x}{\partial x} = f_1 x^2 + f_2 (y^2 - x^2), \quad \frac{\partial \alpha_y}{\partial y} = f_1 y^2 + f_2 (x^2 - y^2), \quad (\text{D.18})$$

with

$$f_1 := \frac{2\kappa_0}{\theta^2} e^{-(\theta/\theta_c)^2}, \quad f_2 := \frac{\kappa_0 \theta_c^2}{\theta^4} \left(1 - e^{-(\theta/\theta_c)^2} \right). \quad (\text{D.19})$$

Finally, the mass reads

$$M(r) = 2\pi \rho_0 r_c^3 \left(\frac{\sqrt{\pi}}{2} \text{erf} \left(\frac{r}{r_c} \right) - \frac{r}{r_c} e^{-(r/r_c)^2} \right) \xrightarrow{r \ll r_c} \frac{4}{3} \pi \rho_0 r^3. \quad (\text{D.20})$$

D.1.3 A possible degeneracy-free MSD model

Bayesian inference pipelines involving MCMC usually struggle in sampling a posterior with degeneracies between sampled parameter. To reduce such an issue for inference involving an approximate core-MSD model, we can implement the MSD model (model 1) of the original model with convergence κ and source position β

$$\kappa_1 = \lambda \kappa + \kappa_c, \quad \beta_1 = \lambda \beta, \quad (\text{D.21})$$

as (model 2)

$$\kappa_2 = \kappa + \lambda^{-1}(\kappa_c - \kappa_0), \quad \beta_2 = \beta, \quad (\text{D.22})$$

where $\kappa_0 := \kappa_c(0)$, with κ_c the convergence of the cored profile, and we set $\lambda = 1 - \kappa_0$. Model 2 is meant to represent a core-MSD where the degeneracy between κ_0 , β and time delays is “subtracted”. Model 1 and model 2, as far as imaging is concerned, are easily seen to be equivalent; in fact, we can write the lens equation from model 1 as (recall that the displacement angle α_c corresponding to a constant convergence κ_0 is $\kappa_0 \theta$)

$$\lambda \beta = \theta - \lambda \alpha - (1 - \lambda) \theta - (\alpha_c - \kappa_0 \theta) \implies \beta = \theta - \alpha - \lambda^{-1}(\alpha_c - \kappa_0 \theta), \quad (\text{D.23})$$

where the last equation is just the lens equation for model 2. The H_0 inference in the two different models are related via $H_0^{(1)} = \lambda H_0^{(2)}$. In fact, from the time delays Eq. (2.49) between image i and j , we can write

$$D_{\text{dt}}^{(1)} = \frac{\Delta T_{ij}}{\lambda} \left(\beta(\theta_i - \theta_j) + \frac{1}{2}(\theta_i^2 - \theta_j^2) + \Psi_j - \Psi_i + \lambda^{-1} \left(\Psi_j^c + \frac{1}{2}\kappa_0\theta_j^2 - \Psi_i^c - \frac{1}{2}\kappa_0\theta_i^2 \right) \right)^{-1}; \quad (\text{D.24})$$

an analogous computation for model 2 yields

$$D_{\text{dt}}^{(2)} = \Delta T_{ij} \frac{D_{\text{dt}}^{(1)} \lambda}{\Delta T_{ij}} = D_{\text{dt}}^{(1)} \lambda \implies H_0^{(1)} = \lambda H_0^{(2)}. \quad (\text{D.25})$$

Notice that model 2 has no degeneracy between κ_0 and H_0 . We remark that model 2 is an artificial model which has a core-MSD without suffering the typical degeneracy of a MSD. The true physical model will be model 1.

One could question why the λ^{-1} factor appears in Eq. (D.22). Suppose truth is model one, and one fits lensing data using $\kappa_{\text{trial}} + \kappa_{\text{c,trial}} - \kappa_{0,\text{trial}}$ (trial here means that one uses the same functional form of model 1 and 2 but without specifying values of parameters), that is a subtracted core model without λ^{-1} factor; then the trial model will converge to $\kappa_{\text{trial}} = \kappa$, $\beta_{\text{trial}} = \beta$ and more importantly $\kappa_{0,\text{trial}} = \lambda^{-1}\kappa_0$, $H_{0,\text{trial}} = H_0^{(2)}$. Hence, one needs to put explicitly the λ^{-1} factor in front of $\kappa_{\text{c,trial}}$, otherwise it will be impossible to recover the value of κ_0 associated to the “true” model 1 (and hence, the H_0 bias coming from such MSD).

D.2 Complex notation lensing and flexion

We here review some key concepts of the complex notation in gravitational lensing, used in Sec. 4.2 [TB23].

D.2.1 The complex notation.

The behavior under rotations of the various lensing quantities acquire a simple form if we use the complex lensing notation [SK95; SE08]. In particular, the complex notation is very convenient in describing the flexion. The complex notation consists in writing 2D vectors like

$$\theta = \theta_1 + i\theta_2, \beta = \beta_1 + i\beta_2 \text{ etc.} \quad (\text{D.26})$$

Then, the lens equation simply reads $\beta = \theta - \alpha$. Under rotations of the coordinate system, these vector quantities transform as $\beta \rightarrow e^{i\phi}\beta$, where ϕ is the angle of the rotation. Quantities which transform like $e^{iN\phi}$ are called spin N quantities. The complex conjugate of a spin N quantity is trivially a spin $-N$ quantity, and in a consistent equation, the spin of the LHS should match the spin of the RHS. The lensing potential Ψ in this formalism is real and a spin 0 quantity. One can obtain the complex deflection angle by means of the derivative operator

$$\nabla_{\text{c}} := \partial_1 + i\partial_2 \implies \alpha = \nabla_{\text{c}}\Psi. \quad (\text{D.27})$$

The operator ∇_{c} increases the spin of the quantity it acts upon by 1; conversely, its complex conjugate ∇_{c}^* reduces the spin by one. It is easy to see that the equations for convergence and shear can be written in this formalism as

$$\kappa = \frac{1}{2}\nabla_{\text{c}}\nabla_{\text{c}}^*\Psi, \quad \gamma = \frac{1}{2}\nabla_{\text{c}}\nabla_{\text{c}}\Psi; \quad (\text{D.28})$$

notice the spin-2 behavior of the shear. Moreover, we can express third order derivatives of the lensing potential as derivatives of the shear,

$$G = \nabla_c \gamma, \quad F = \nabla_c^* \gamma, \quad (\text{D.29})$$

where F and G are the flexion [GB05; Bac+06]. Explicitly,

$$F_1 = \frac{\partial_1^3 + \partial_1 \partial_2^2}{2} \Psi, \quad F_2 = \frac{\partial_2^3 + \partial_1^2 \partial_2}{2} \Psi, \quad G_1 = \frac{\partial_1^3 - 3\partial_1 \partial_2^2}{2} \Psi, \quad G_2 = \frac{-\partial_2^3 + 3\partial_1^2 \partial_2}{2} \Psi. \quad (\text{D.30})$$

For an axisymmetric profile, writing the angle $\theta = |\theta|e^{i\phi}$, we can write the complex shear as

$$\gamma = \Gamma(|\theta|)e^{2i\phi}, \quad (\text{D.31})$$

where $\Gamma(|\theta|)$ is a real function depending on the modulus of θ . Written in this way, the spin-2 character of the shear is manifest. For axisymmetric profiles, also the flexion quantities acquire a simple form, which reads

$$G(|\theta|, \phi) = e^{3i\phi} \left(\frac{d\Gamma}{d|\theta|} - \frac{2\Gamma}{|\theta|} \right), \quad F(|\theta|, \phi) = e^{i\phi} \left(\frac{d\Gamma}{d|\theta|} + \frac{2\Gamma}{|\theta|} \right). \quad (\text{D.32})$$

D.2.2 Complex notation for flexion degeneracies

In Sec. 4.2, we described a discrete degeneracy involving a NFW halo of the group where the lens resides. In this section, we work out the formulas for convergence, shear and flexion of an halo displaced from the main lens position.

The lensing potential associated to a NFW profile can be written as

$$\Psi_{\text{NFW}}(\theta) = 2\rho_\theta \theta_s^3 \left(\log^2 \left(\frac{\theta}{2\theta_s} \right) - \text{arccosh}^2 \left(\frac{\theta_s}{\theta} \right) \right), \quad \theta_s := R_s/D_1, \quad \rho_\theta := \frac{\rho_s D_1}{\Sigma_{\text{cr}}}. \quad (\text{D.33})$$

The previous is formally valid only for $\theta < \theta_s$, but its validity can be extended to $\theta > \theta_s$ if one makes use of logarithmic definition of $\text{arccosh}(z) = \ln(z + \sqrt{z^2 - 1})$ and do computations with complex numbers (the final result will end up being real). The lens potential for a displaced NFW halo, with $\vec{h}$ as the center of coordinates of the displaced halo with respect to the main lens center, reads

$$\Psi_{\text{halo}}(\theta_1, \theta_2) := \Psi_{\text{NFW}}(|\vec{\theta} - \vec{h}|) = \Psi_{\text{NFW}} \left(h \sqrt{\left(\frac{\theta_1}{|h|} \right)^2 + \left(\frac{\theta_2}{|h|} \right)^2 + 1 - 2 \frac{\theta_1 h_1 + \theta_2 h_2}{|h|}} \right). \quad (\text{D.34})$$

Assuming the typical angular scale θ where images form to be much smaller than the displacement size $|h|$, we can expand the previous with respect to the small parameters $\theta_1/|h|$, $\theta_2/|h|$ around $\theta_1/|h| = \theta_2/|h| = 0$; notice that this expansion is mathematically equivalent to an expansion for small θ_1 , θ_2 ,

$$\Psi_{\text{halo}}(\theta_1, \theta_2) \simeq \Psi_{\text{halo}}(0, 0) + \nabla_{\vec{\theta}/|h|} \Psi_{\text{halo}}(0, 0) \cdot \frac{\vec{\theta}}{|h|} + \dots = \Psi_{\text{halo}}(0, 0) + \nabla_{\vec{\theta}} \Psi_{\text{halo}}(0, 0) \cdot \vec{\theta} + \dots \quad (\text{D.35})$$

Recall that the lensing potential and the deflection angle are related as $\nabla \Psi = \vec{\alpha}$, Eq. (2.27); in particular, we can write the lens equation, including the potential Ψ_{lens} of the main lens, as

$$\beta_i = \theta_i - \partial_i \Psi_{\text{lens}}(\vec{\theta}) - \partial_i \Psi_{\text{halo}}(0, 0) - \partial_i \partial_j \Psi_{\text{halo}}(0, 0) \theta_j + \dots \quad (\text{D.36})$$

The zeroth order of the expansion of the halo displacement angle, $\partial_i \Psi_{\text{halo}}(0, 0)$, is responsible of the source-halo position degeneracy (since it is a constant term); the first order term can be seen as an

external convergence term (not affecting imaging) and an external shear term,

$$\partial_i \partial_j \Psi_{\text{nfw}}(0,0) \theta_j = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \vec{\theta}, \quad \kappa = \frac{1}{2}(\Psi_{11} + \Psi_{22}), \quad \gamma_1 = \frac{1}{2}(\Psi_{11} - \Psi_{22}), \quad \gamma_2 = \Psi_{12}. \quad (\text{D.37})$$

In complex notation, the lens equation up to third order in the expansion of Ψ_{halo} can be written as

$$\beta + \nabla_c \Psi_{\text{halo}}(0) = \theta - \nabla_c \Psi_{\text{lens}}(\theta) - \kappa \theta - (\gamma + \gamma^{\text{ext}}) \theta^* - \frac{1}{4} F^* \theta^2 - \frac{1}{2} F \theta \theta^* - \frac{1}{4} G (\theta^*)^2, \quad (\text{D.38})$$

where we included the contribution of the external shear γ^{ext} , which would take into account contributions from large scale structures on the line of sight to the lens and source system.

Axisymmetric system. If Ψ_{NFW} is assumed to be spherical (i.e. without ellipticity), then we can use cylindrical coordinates¹

$$\vec{h} - \vec{\theta} = r(\cos \phi, \sin \phi)^\top; \quad (\text{D.39})$$

then, we can write

$$\frac{\partial}{\partial \theta_i} \Psi_{\text{halo}} = \frac{\partial r}{\partial \theta_i} \frac{\partial}{\partial r} \Psi_{\text{NFW}}(r) = \frac{\theta_i - h_i}{r} \Psi'_{\text{NFW}}(r); \quad (\text{D.40})$$

in complex notation, it translates into

$$\nabla_c \Psi_{\text{halo}} = -e^{i\phi} \Psi'_{\text{NFW}}(r); \quad (\text{D.41})$$

notice that

$$\frac{\partial \phi}{\partial \theta_1} = \frac{\sin \phi}{r}, \quad \frac{\partial \phi}{\partial \theta_2} = -\frac{\cos \phi}{r}, \quad (\text{D.42})$$

which have opposite sign with respect to an usual choice of cylindrical coordinate, due to our choice in Eq. (D.39). We can thus write

$$\nabla_c = -e^{i\phi} \left(\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right), \quad (\text{D.43})$$

so that

$$\begin{aligned} \kappa &= \frac{1}{2} \left(\Psi''_{\text{NFW}} + \frac{\Psi'_{\text{NFW}}}{r} \right), \quad \gamma = \frac{e^{2i\phi}}{2} \left(\Psi''_{\text{NFW}} - \frac{\Psi'_{\text{NFW}}}{r} \right), \\ G &= -\frac{e^{3i\phi}}{2} \left(\Psi'''_{\text{NFW}} - 3 \frac{\Psi''_{\text{NFW}}}{r} + 3 \frac{\Psi'_{\text{NFW}}}{r^2} \right), \\ F &= -\frac{e^{i\phi}}{2} \left(\Psi'''_{\text{NFW}} + \frac{\Psi''_{\text{NFW}}}{r} - \frac{\Psi'_{\text{NFW}}}{r^2} \right) = -e^{i\phi} \frac{d\kappa}{dr}. \end{aligned} \quad (\text{D.44})$$

¹With this definition, when $\theta = 0$, ϕ corresponds to the angle between the origin and the position of the halo center; this will introduce minus sign differences with respect to what one could naively expect.

Explicitly, we have (for notation simplicity, from now on $h \equiv |h|$)

$$\nabla_c \Psi_{\text{halo}}(0) = -4\rho_\theta \frac{\theta_s^3}{h} \left(\frac{\text{arccosh}(\theta_s/h)}{\sqrt{1-h^2/\theta_s^2}} + \log \frac{h}{2\theta_s} \right) e^{i\phi}, \quad (\text{D.45})$$

$$\kappa = \frac{2\rho_\theta \theta_s^3}{h^2 - \theta_s^2} \left(1 - \frac{\text{arccosh}(\theta_s/h)}{\sqrt{1-h^2/\theta_s^2}} \right), \quad (\text{D.46})$$

$$\Gamma(0) = \frac{2\rho_\theta \theta_s^3}{h^2 - \theta_s^2} \left[1 + \frac{(2\theta_s^2 - 3h^2) \text{arccosh}(\theta_s/h)}{h^2 \sqrt{1-h^2/\theta_s^2}} - 2 \left(1 - \left(\frac{\theta_s}{h} \right)^2 \right) \log \left(\frac{h}{2\theta_s} \right) \right], \quad (\text{D.47})$$

$$F(0,0) = \frac{2\rho_\theta \theta_s^3}{h(h^2 - \theta_s^2)^2} \left(2h^2 + \theta_s^2 - \frac{3h^2 \text{arccosh}(\theta_s/h)}{\sqrt{1-h^2/\theta_s^2}} \right), \quad (\text{D.48})$$

$$G(0,0) = -2\rho_\theta \theta_s^3 \frac{\theta_s(15h^4 - 20\theta_s^2 h^2 + 8\theta_s^4) \text{arccosh}(\theta_s/h)}{h^3(h^2 - \theta_s^2)^2 \sqrt{\theta_s^2 - h^2}} \quad (\text{D.49})$$

$$+ \frac{\sqrt{\theta_s^2 - h^2}(-6h^4 + 3h^2 \theta_s^2 + 8(h^2 - \theta_s^2)^2 \log(h/2\theta_s))}{h^3(h^2 - \theta_s^2)^2 \sqrt{\theta_s^2 - h^2}}. \quad (\text{D.50})$$

For reference, if the halo would have had a power law profile Eq. (2.41) (i.e. assuming we can approximate NFW with a power law in the region of interest), we have

$$\nabla_c \Psi_{\text{halo}}(0) = -\left(\frac{\theta_E}{h} \right)^{\gamma-1} h e^{i\phi}, \quad (\text{D.51})$$

$$\kappa = -\frac{1}{2}(\gamma - 3) \left(\frac{\theta_E}{h} \right)^{\gamma-1}, \quad (\text{D.52})$$

$$\Gamma(0) = -\frac{1}{2}(\gamma - 1) \left(\frac{\theta_E}{h} \right)^{\gamma-1}, \quad (\text{D.53})$$

$$F(0,0) = -\frac{1}{2h}(\gamma - 3)(\gamma - 1) \left(\frac{\theta_E}{h} \right)^{\gamma-1}, \quad (\text{D.54})$$

$$G(0,0) = -\frac{1}{2h}(\gamma^2 - 1) \left(\frac{\theta_E}{h} \right)^{\gamma-1}. \quad (\text{D.55})$$

Inserting ellipticity

We can implement ellipticity in NFW profiles by using pseudo-elliptical NFW lens models [GK02; DMCM12], which is a reliable approximation as long as the ellipticity parameter is not too large.

This approximation involves using the spherical NFW lensing potential, but computed in elliptical coordinates²,

$$\Psi_\epsilon(\vec{\theta}) = \Psi(\vec{\theta}_\epsilon), \quad \vec{\theta}_\epsilon = \begin{pmatrix} \cos \varphi \sqrt{1-\epsilon} & \sin \varphi \sqrt{1-\epsilon} \\ -\sin \varphi \sqrt{1+\epsilon} & \cos \varphi \sqrt{1+\epsilon} \end{pmatrix} \vec{\theta}, \quad (\text{D.56})$$

where

$$\epsilon = \frac{2e}{1+e^2}, \quad e = \sqrt{e_1^2 + e_2^2}, \quad \varphi = \frac{1}{2} \arctan \frac{e_2}{e_1}. \quad (\text{D.57})$$

Notice that under rotations of the coordinate system, ϵ is invariant but φ changes. The expansion of the potential $\Psi_{\text{halo},\epsilon}$ follows Eq. (D.35); however, since the potential is not axisymmetric, it is not possible to factor out the ϕ dependence on γ_ϵ , G_ϵ , F_ϵ as we did previously. Nevertheless, it is possible to express

²This is how it is implemented in `lenstronomy`

these quantities as functions of $\gamma_0 := \Gamma(0)$, $F_0 := F(0,0)$ and $G_0 := G(0,0)$. In the following, we will sketch how this can be done.

Notice that we can write (here, $\vec{\theta} = (x, y)^\top$)

$$\nabla_c = e^{i\varphi} \left(\sqrt{1-\epsilon} \frac{\partial}{\partial x_\epsilon} + i\sqrt{1+\epsilon} \frac{\partial}{\partial y_\epsilon} \right); \quad (D.58)$$

as an example, we will take

$$\gamma_\epsilon = \frac{1}{2} \nabla_c \nabla_c \Psi_\epsilon = \frac{e^{2i\varphi}}{2} \left((1-\epsilon) \frac{\partial^2}{\partial x_\epsilon^2} - (1+\epsilon) \frac{\partial^2}{\partial y_\epsilon^2} + 2i\sqrt{1-\epsilon^2} \frac{\partial}{\partial x_\epsilon} \frac{\partial}{\partial y_\epsilon} \right) \Psi_\epsilon, \quad (D.59)$$

in particular

$$\gamma_\epsilon = e^{2i\varphi} (\gamma_1(\vec{\theta}_\epsilon) + i\sqrt{1-\epsilon^2} \gamma_2(\vec{\theta}_\epsilon) - \epsilon \kappa(\vec{\theta}_\epsilon)), \quad (D.60)$$

where the quantities without ϵ subscript refer to the spherical case quantities, in this case computed at the elliptic coordinates. By defining $\vec{h} \leftrightarrow h_\epsilon e^{i\phi_\epsilon}$, i.e. the displacement vector $\vec{h}$ in elliptical coordinates in the complex notation, we can express

$$\gamma = \gamma_0(h_\epsilon(h, \phi)) e^{i\phi_\epsilon(\phi)}, \quad (D.61)$$

where γ_0 is the spherical expression. Notice that $\phi_\epsilon \sim \phi - \varphi$. For κ , one has

$$\kappa_\epsilon = \kappa(h_\epsilon(h, \phi)) - \epsilon \gamma_1(h_\epsilon(h, \phi)). \quad (D.62)$$

A similar reasoning follows for F_ϵ, G_ϵ :

$$\begin{aligned} F_\epsilon &= \frac{e^{i\varphi}}{2} \left((1-\epsilon)^{3/2} \partial_{x_\epsilon}^3 + \sqrt{1-\epsilon} (1+\epsilon) \partial_{x_\epsilon} \partial_{y_\epsilon}^2 + i((1+\epsilon)^{3/2} \partial_{y_\epsilon}^3 + \sqrt{1+\epsilon} (1-\epsilon) \partial_{x_\epsilon}^2 \partial_{y_\epsilon}) \right) \Psi, \\ G_\epsilon &= \frac{e^{3i\varphi}}{2} \left((1-\epsilon)^{3/2} \partial_{x_\epsilon}^3 - 3\sqrt{1-\epsilon} (1+\epsilon) \partial_{x_\epsilon} \partial_{y_\epsilon}^2 + i(-(1+\epsilon)^{3/2} \partial_{y_\epsilon}^3 + 3\sqrt{1+\epsilon} (1-\epsilon) \partial_{x_\epsilon}^2 \partial_{y_\epsilon}) \right) \Psi. \end{aligned} \quad (D.63)$$

To try to have more insightful expression, we can expand at first order the ϵ factors outside derivatives and obtain

$$\begin{aligned} F_\epsilon &\simeq e^{i(\varphi+\phi_\epsilon)} F_0(h_\epsilon) \left(1 - \frac{\epsilon}{2} \left(2e^{-2i\phi_\epsilon} + e^{2i\phi_\epsilon} \frac{G_0(h_\epsilon)}{F_0(h_\epsilon)} \right) \right), \\ G_\epsilon &\simeq e^{3i(\varphi+\phi_\epsilon)} G_0(h_\epsilon) \left(1 - \frac{3\epsilon}{2} e^{-2i\phi_\epsilon} \frac{F_0(h_\epsilon)}{G_0(h_\epsilon)} \right). \end{aligned} \quad (D.64)$$

Time delays

One can find the time delay from the complex lens equation, by noticing (thanks to the chain rule)

$$\nabla_c = 2 \frac{\partial}{\partial \theta^*} \implies T(\theta) \propto \frac{1}{2} \int d\theta^* \alpha + f(\theta), \quad (D.65)$$

where the integral is a definite integral, and θ is the complex angle corresponding to an image. Notice that the integral misses a possible function of θ alone, other than a proportionality factor. This function can be recovered by imposing the time delay to be real, $T = T^*$.³

³Notice that $\Delta T = \Delta T(\theta, \theta^*)$, so the usual complex integration and differentiation cannot be used, since the time delay does not depend on θ alone.

Exploiting Eq. (D.65) using the complex deflection angle expansion in Eq. (D.38), we can write

$$T(\theta_A) \propto \frac{\theta_A \theta_A^*}{2} - \frac{1}{2}[(\beta + \nabla_c \Psi_{\text{nfw}})\theta_A^* + \text{c.c.}] - \Psi_{\text{lens}}(\vec{\theta}_A) - \Psi_{\text{halo}}(0,0) - \frac{1}{2}\kappa\theta_A\theta_A^* - \frac{1}{4}[(\gamma + \gamma^{\text{ext}})\theta_A^{*2} + \text{c.c.}] - \frac{1}{24}[3F\theta_A^{*2}\theta_A + G\theta_A^{*3} + \text{c.c.}] . \quad (\text{D.66})$$

In real notation,

$$T(\theta_A) \propto \frac{\theta_A^2}{2} - (\beta_i + \partial_i \Psi_{\text{nfw}})\theta_i^A - \Psi_{\text{lens}}(\vec{\theta}_A) - \Psi_{\text{halo}}(0,0) - \frac{1}{2}\kappa\theta_A^2 - \frac{1}{2}(\gamma_1 + \gamma_1^{\text{ext}})(\theta_1^A{}^2 - \theta_2^A{}^2) - (\gamma_2 + \gamma_2^{\text{ext}})\theta_1^A\theta_2^A - \frac{1}{12}\left((3F_1 + G_1)\theta_1^A{}^3 + (3F_2 - G_2)\theta_2^A{}^3 + 3(F_1 - G_1)\theta_1^A\theta_2^A{}^2 + 3(F_2 + G_2)\theta_2^A\theta_1^A{}^2\right) . \quad (\text{D.67})$$

The proportionality constant cannot be obtain via this method alone, which relies on Fermat principle, but we know by direct comparison with the usual result Eq. (2.49) that such proportionality constant should be D_{dt} . Notice that the term $\Psi_{\text{halo}}(0,0)$ is a constant which will cancel in the difference between two images, hence we can ignore it.

Higher orders. It is possible to straightforwardly generalize the expansion of the complex deflection angle in Eq. (D.38) to orders higher than the flexion. It is sufficient to notice that, for a function of two variables, we can write its Taylor expansion around zero as

$$\Psi(x, y) = \sum_{n=0}^{\infty} \frac{1}{n!} (x\partial_x + y\partial_y)^n \Psi(0,0) = \sum_{n=0}^{\infty} \frac{1}{n!2^n} (\theta^* \nabla_c + \theta \nabla_c^*)^n \Psi(0,0) , \quad (\text{D.68})$$

where $\theta = x + iy$. Then, the displacement angle corresponding to Ψ would be obtained by:

$$\alpha = 2 \frac{\partial \Psi}{\partial \theta^*} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{\theta^* \nabla_c + \theta \nabla_c^*}{2} \right)^{n-1} \nabla_c \Psi(0,0) . \quad (\text{D.69})$$

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